Spontaneous Scalarization of Black Holes in Gauss-Bonnet Teleparallel Gravity

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Canadian Quantum Research Center;
arXiv:2212.07653 and our review Rept. Prog. Phys. **86** (2023) no.2, 026901

Overview of the Talk

1. Introduction to Teleparallel theories of gravity
   - Basic mathematical ingredients
   - Trinity of gravity
2. Black hole hair and Riemannian extensions of GR
   - No-hair theorem and a scalar field theory non-minimally coupled to the Ricci scalar
   - Scalar-Gauss Bonnet gravity
3. Teleparallel scalar Gauss-Bonnet gravity
   - Scalar field non-minimally coupled to Torsion
   - Teleparallel scalar Gauss-Bonnet

Fundamental variables and characteristic tensors

- In the most general metric-affine setting, the fundamental variables are a **metric** $g_{\mu\nu}$ (10 comp.) as well as the coefficients $\hat{\Gamma}^{\rho}{}_{\mu\nu}$ (64 comp.) of an **affine connection**.

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Curvature	$\tilde{R}^{\mu}{}_{\nu\rho\sigma} = \partial_{\rho}\tilde{\Gamma}^{\mu}{}_{\nu\sigma} - \partial_{\sigma}\tilde{\Gamma}^{\mu}{}_{\nu\rho} + \tilde{\Gamma}^{\mu}{}_{\tau\rho}\tilde{\Gamma}^{\tau}{}_{\nu\sigma} - \tilde{\Gamma}^{\mu}{}_{\tau\sigma}\tilde{\Gamma}^{\tau}{}_{\nu\rho}$
Torsion	$\tilde{T}^{\mu}{}_{\nu\rho} = \tilde{\Gamma}^{\mu}{}_{\rho\nu} - \tilde{\Gamma}^{\mu}{}_{\nu\rho}$
Nonmetricity	$\tilde{Q}_{\mu\nu\rho} = \tilde{\nabla}_{\mu}g_{\nu\rho} = \partial_{\mu}g_{\nu\rho} - \tilde{\Gamma}^{\sigma}{}_{\nu\mu}g_{\sigma\rho} - \tilde{\Gamma}^{\sigma}{}_{\rho\mu}g_{\nu\sigma}$

Tetrads and spin connection

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where η_{ab} is the Minkowski metric.

- Quantities denoted with a circle on top $\circ$ denote that they are defined with respect to the Levi-Civita connection and hats are general affine connection.

Trinity of gravity - curvature tensor

- The curvature becomes

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with

$$T := T^{\rho\lambda\kappa}T_{\rho\lambda\kappa} + 2T^{\rho\lambda\kappa}T_{\kappa\rho\lambda} - 4T_{\rho}{}^{\kappa}{}_{\kappa}T^{\rho\lambda}{}_{\lambda}\,, \quad \text{Torsion scalar}\,,$$

$$Q := -\frac{1}{4}Q_{\alpha\beta\gamma}Q^{\alpha\beta\gamma} + \frac{1}{2}Q_{\alpha\beta\gamma}Q^{\beta\alpha\gamma} + \frac{1}{4}Q_{\alpha}Q^{\alpha} - \frac{1}{2}Q_{\alpha}\bar{Q}^{\alpha}\,, \ \text{Nonmetricity scalar}\,,$$

$$C := 2(Q_{\kappa\rho\lambda}T^{\lambda\kappa\rho} + Q_{\rho}{}^{\sigma}{}_{\sigma}T^{\rho\kappa}{}_{\kappa} - Q^{\sigma}{}_{\sigma\rho}T^{\rho\kappa}{}_{\kappa})\,.$$

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where $\kappa^2 = 8\pi G$ and $L_{\rm m}$ is any matter Lagrangian.

- The Einstein's field equations are obtained by taking variations w/r to the metric: $\boxed{\mathring{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathring{R} = \kappa^2 T_{\mu\nu}}$.

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(torsional) Teleparallel equivalent of GR (TEGR) action

$$S_{\rm TEGR} = \int \left[-\frac{1}{2\kappa^2}T + L_{\rm m}\right] e\, d^4x\,.$$

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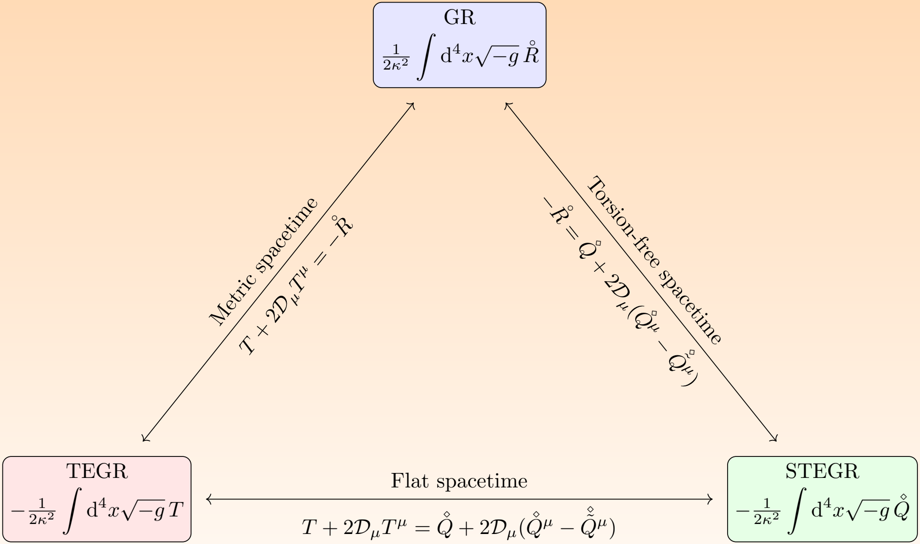

Figure: Geometrical trinity of gravity (S. Bahamonde et.al., “Teleparallel Gravity: From Theory o Cosmology,” Rept. Prog. Phys. **86** (2023) no.2, 026901.; J. Beltrán Jiménez, L. Heisenberg and T. S. Koivisto, “The Geometrical Trinity of Gravity,” Universe **5** (2019) no.7, 173.)

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- This is a theorem, meaning that in GR, this is the unique asymptotically flat black hole solution. In spherical symmetry, it is just Schwarzschild.
- Is it possible that black holes have hair? One needs to go beyond GR, either by having modified gravity or allowing extra degrees of freedom such as scalar fields coupled to gravity.

Scalar fields non-minimally coupled to the Ricci scalar

- Let us take the following simple extension of GR by allowing couplings between the Ricci scalar and a scalar field with its kinetic term X and a potential:

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- It seems that the simple theory fails to have more general realistic black holes (with hair).
- Is it then not possible to have something realistic beyond Kerr?

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- One can then modify the theory by having new terms in the action. New possible curvature invariants can be added and the first ones that one could think are quadratic contractions of curvature.

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- However, if one allows couplings between the Gauss-Bonnet invariant and a scalar field, then the field equations will not be longer equivalent to GR.

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- The Scalar Gauss-Bonnet (sGB) gravity theory is described by the following action

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- Note: $\mathring{R} = 0$ in Schwarzschild but $\mathring{G} \neq 0$. That property would be important to understand the difference between this model and the former one.

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- For strong gravitational fields at the horizon, that realizes when the black hole mass M falls below a certain threshold, the Schwarzschild solution becomes unstable and a non-trivial scalar field emerges.
- Thus the solution bifurcates to a different black hole solution with scalar hair.
- This transition is usually smooth in sGB and shares similarities with second order phase transitions.

Scalar Gauss-Bonnet and Spontaneous scalarization

- The simplest way to study the spontaneous scalarization process is by considering deviations from the Schwarzschild metric by choosing the metric to have the form

$$ds^2 = e^{\delta(r)}A(r)dt^2 - \tfrac{1}{A(r)}dr^2 - r^2 d\Omega^2\,, \quad \delta(r) \ll 1\,, \quad A(r) = 1 - \frac{2M}{r}\,.$$

Then, one takes perturbations of the scalar field that can be written as

$$\delta\psi(t,r,\theta,\psi) = \tfrac{u(r)}{r}\, e^{-i\omega t}\, Y_{lm}(\theta,\psi)\,,$$

which allows us to decouple the metric field equations from the scalar field equations.

Scalar Gauss-Bonnet and Spontaneous scalarization

- By plugging those expressions into the scalar field equation ($\beta\mathring{\Box}\psi + \alpha\dot{\mathcal{G}}(\psi)\mathring{G} = 0$) and after expanding them up to first order, we obtain the following Schrodinger-like form equation

$$\frac{d^2u}{dr_*^2} + [\omega^2 - U(r)]u = 0\,, \quad \text{with} \quad \mathcal{G}'(\psi_0) = 0\,,$$

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  where we have introduced tortoise coordinates $dr_* = (1-2M/r)^{-1}dr$ and the Potential $U(r)$ is given by

$$U(r) = \Big(1 - \frac{2M}{r}\Big)\Big[\frac{2M}{r^3} + \frac{l(l+1)}{r^2} + \frac{48M^2}{r^6\beta}\alpha\ddot{\mathcal{G}}(\psi_0)\Big]\,.$$

- Here ψ_0 is assumed to be the constant Schwarzschild geometry background value of the scalar field.

- A sufficient condition for having an unstable mode is

$$\int_{-\infty}^{+\infty} U(r_*)dr_* = \int_{2M}^{\infty} \frac{U(r)}{1-\frac{2M}{r}}dr < 0\,.$$

Therefore, the theory gives us the possibility to have such an unstable mode if

$$\frac{3\alpha\ddot{\mathcal{G}}(\psi_0)+5\beta M^2}{10\beta M^3} < 0\,.$$

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- Then, for weak field regime the scalar charge does not emerge but after passing a certain threshold which is related to strong gravity regime (given by the above inequality), the Schwarzschild solution is unstable and then, the charge emerges and the solution is not longer Schwarzschild.

Black holes in Scalar Gauss-Bonnet

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- Usually, exponential and power-law couplings give those solutions.
- It has been proved that those solutions (numerically) are stable against linear perturbations.
- The study has been generalized for rotating solutions, charged black holes or even multi-scalar field configurations.

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- What about non-Riemannian geometry?

Overview of the Talk

- In our previous paper[1], we studied Teleparallel theories with a scalar field, for example[2]:

$$S = \frac{1}{2\kappa^2}\int_M \left[-\mathcal{A}(\psi)T - \tilde{\mathcal{C}}(\psi)B - \frac{1}{2}\mathcal{B}(\psi)\partial_\mu\psi\partial^\mu\psi - 2\kappa^2\mathcal{V}(\psi)\right]\sqrt{-g}\,\mathrm{d}^4x\,,$$

where $X = -\frac{1}{2}g^{\mu\nu}\partial_\mu\psi\partial_\nu\psi$.

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- Since $\overset{\circ}{R} = -T + B$, when $\mathcal{A}(\psi) = -\tilde{\mathcal{C}}(\psi)$ the above theory is exactly the same as the standard non-minimally one.

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- We found new exact black hole solutions (some of them different to the Riemannian case), but they seem to be not so much interesting phenomenologically.

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Here, D_λ is the cov derivative of the general connection.

- Then, two Teleparallel Gauss-Bonnet invariants appear in the Teleparallel framework. T_G is a topological invariant in $4D$ and B_G is a boundary term (in all dimensions).

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$$\mathcal{S}_{\rm TsGB} = \frac{1}{2\kappa^2}\int\Big[-T - \frac{1}{2}\beta\,\partial_\mu\psi\partial^\mu\psi + \alpha_1\mathcal{G}_1(\psi)T_G + \alpha_2\mathcal{G}_2(\psi)B_G\Big]\,e\,\mathrm{d}^4x\,,$$

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- For other coupling cases, the theory is different from the Riemannian case.

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- It is convenient to re-parametrize the action such that one has the Riemannian case: (Note again $\mathring{G} = T_G + B_G$)

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$$= \frac{1}{2\kappa^2}\int \Big[\mathring{R} - \frac{1}{2}\beta\,\partial_\mu\psi\partial^\mu\psi + \alpha_2\mathcal{G}_2(\psi)\mathring{G} + \alpha_3\mathcal{G}_3(\psi)T_G \Big]\, e\,\mathrm{d}^4x\,,$$

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- The second and third cases ($\alpha_3 \neq 0$) are new in the literature and they can only exist when one considers Teleparallel gravity.

Spherical Symmetry in Teleparallel sGB

- In TG, the dynamical variable is the tetrad field. The most general tetrad satisfying spherical symmetry in the Weitzenbock gauge is

$$e^{A}{}_{\nu} = \begin{pmatrix} C_1 & C_2 & 0 & 0 \\ C_3 \sin\theta\cos\phi & C_4 \sin\theta\cos\phi & C_5 \cos\theta\cos\phi - C_6 \sin\phi & -\sin\theta(C_5 \sin\phi + C_6 \cos\theta\cos\phi) \\ C_3 \sin\theta\sin\phi & C_4 \sin\theta\sin\phi & C_5 \cos\theta\sin\phi + C_6 \cos\phi & \sin\theta(C_5 \cos\phi - C_6 \cos\theta\sin\phi) \\ C_3 \cos\theta & C_4 \cos\theta & -C_5 \sin\theta & C_6 \sin^2\theta \end{pmatrix},$$

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  where $C_i = C_i(t,r)$ but hereafter, we will consider the stationary case $C_i = C_i(r)$.
- Using $g_{\mu\nu} = \eta_{AB}e^A{}_\mu e^B{}_\nu$, we have that the metric is

$$\begin{aligned} \mathrm{d}s^2 &= \left(C_1^2 - C_3^2\right)\mathrm{d}t^2 - 2(C_3C_4 - C_1C_2)\,\mathrm{d}t\,\mathrm{d}r - \left(C_4^2 - C_2^2\right)\mathrm{d}r^2 \\ &\quad - \left(C_5^2 + C_6^2\right)\left(\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2\right), \end{aligned}$$

  where we have cross-terms.

Spherical Symmetry in Teleparallel sGB

- Without losing generality, we can choose a coordinate system such that the cross term vanishes. This can be easily done by taking the following reparametrization:

$$\begin{aligned} C_1(r) &= \nu A(r) \cosh \beta(r)\,, \quad C_3(r) = \nu A(r) \sinh \beta(r)\,, \\ C_4(r) &= \xi B(r) \cosh \beta(r)\,, \quad C_2(r) = \xi B(r) \sinh \beta(r)\,, \\ C_5(r) &= \chi C(r) \cos \alpha(r)\,, \quad C_6(r) = \chi C(r) \sin \alpha(r)\,, \end{aligned}$$

with $\{\nu, \xi, \chi\}$ being ± 1. This tetrad gives the metric in the standard form in spherical coordinates:

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- Note that $\beta(r), \alpha(r)$ are tetrad dof (they do not appear in the metric). They can be set by solving the antisymmetric field equations.

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- The first branch which solves the antisymmetric equations is $\beta(r) = i\pi\, n_1\,, \alpha(r) = \pi\, n_2$ which gives

$$e^{(1)\,a}{}_{\mu} = \begin{pmatrix} \nu A & 0 & 0 & 0 \\ 0 & \xi B \sin\theta\cos\phi & \chi C\cos\theta\cos\phi & -\chi C\sin\theta\sin\phi \\ 0 & \xi B\sin\theta\sin\phi & \chi C\cos\theta\sin\phi & \chi C\sin\theta\cos\phi \\ 0 & \xi B\cos\theta & -\chi C\sin\theta & 0 \end{pmatrix}, \quad \{\nu,\xi,\chi\} = \pm 1\,.$$

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- The second branch branch which solves the antisymmetric equations is $\beta(r) = \frac{i\pi}{2} + i\pi\, n_3 \,, \alpha(r) = \frac{\pi}{2} + \pi$ which gives

$$e^{(2)\,a}{}_{\mu} = \begin{pmatrix} 0 & i\xi B & 0 & 0 \\ i\nu A\sin\theta\cos\phi & 0 & -\chi C\sin\phi & -\chi C\sin\theta\cos\theta\cos\phi \\ i\nu A\sin\theta\sin\phi & 0 & \chi C\cos\phi & -\chi C\sin\theta\cos\theta\sin\phi \\ i\nu A\cos\theta & 0 & 0 & \chi C\sin^2\theta \end{pmatrix}\,, \quad \{\nu,\xi,\chi\} = \pm 1\,.$$

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- One example of this would be in GR in axial symmetry. Kerr is not the unique axial solution. We also for example have the so-called Taub-NUT solution which is axially symmetric but it is not asymptotically flat so that, it cannot describe realistic astrophysical black hole configurations.
- In our paper, we focused on the complex tetrad $e^{(2)a}{}_{\mu}$ since in previous papers, we have found exact BH solutions (such in $f(T)$ Born-Infeld gravity).

- We can follow a similar computation as in the sGB case to arrive at

$$\frac{d^2u}{dr_*^2} + [\omega^2 - U(r)]u = 0\,, \quad \text{with} \quad \mathcal{G}_i'(\psi_0) = 0\,,$$

and now the potential is more general:

$$U(r) = \left(1 - \frac{2M}{r}\right)\Big[\frac{2M}{r^3} + \frac{l(l+1)}{r^2} - \frac{32M}{r^5\beta}\alpha_3\ddot{\mathcal{G}}_3(\psi_0) + \frac{48M^2}{r^6\beta}(\alpha_3\ddot{\mathcal{G}}_3(\psi_0) + \alpha_2\ddot{\mathcal{G}}_2(\psi_0))\Big]\,.$$

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- The theory gives us the possibility to have such an unstable mode if

$$\frac{-4\alpha_3\ddot{\mathcal{G}}_3(\psi_0)+6\alpha_2\ddot{\mathcal{G}}_2(\psi_0)+5\beta M^2}{20\beta M^3}<0\,.$$

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- For our theory, spontaneous scalarization can occur in our theory for a much larger choice of parameters for different masses.
- Non-rotating black holes was possible only for $\alpha_2<0$ in sGB.

- We can find analytical perturbed solutions around Schwarzschild that can be obtained by taking

$$
\begin{aligned}
A(r)^2 &= 1-\frac{2M}{r}+\epsilon\, a_1(r)+\epsilon^2\, a_2(r)\,,\\
B(r)^{-2} &= 1-\frac{2M}{r}+\epsilon\, b_1(r)+\epsilon^2\, b_2(r)\,,\\
\alpha_i\mathcal{G}_i(\psi) &= \epsilon\,\alpha_i\,\mathcal{G}_i(\psi_\infty)+\frac{\epsilon\,\alpha_i}{M^2}\mathcal{G}_i'(\psi_\infty)(\psi-\psi_\infty)\\
&\quad+\frac{\epsilon\,\alpha_i}{2M^4}\mathcal{G}_i''(\psi_\infty)(\psi-\psi_\infty)^2\,,\\
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  where $\epsilon \ll 1$ is a small tracking parameter.
- We found a solution of the theory perturbatly which represents a scalar-hair-endowed Schwarzschild black hole which is a generalization of the well-known sGB solution presented in previous Riemannian papers.

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- Let us now take expansions near the horizon r_H as

$$\begin{aligned} A(r)^2 &= a_1(r-r_H)+a_2(r-r_H)^2+\dots\,, \\ B(r)^{-2} &= b_1(r-r_H)+b_2(r-r_H)^2+\dots\,, \\ \psi(r) &= \psi_H+\psi'_H(r-r_H)+\psi''_H(r-r_H)^2+\dots\,. \end{aligned}$$

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- One can easily solve the system near the horizon and find two branches of coupling.
- When $\alpha_3\dot{\mathcal{G}}_3=\alpha_2\dot{\mathcal{G}}_2$, the scalar field must satisfy

$$\psi'_H=\frac{8\alpha_2\dot{\mathcal{G}}_2(\psi_H)}{\beta r_H^3}\,.$$

- When $\alpha_3\dot{\mathcal{G}}_3 \neq \alpha_2\dot{\mathcal{G}}_2$, the scalar field at the horizon must satisfy

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- This branch gives the correct condition for the Riemannian sGB case ($\alpha_3 = 0$) that has been used widely in the literature to solving the equations numerically:

$$\psi'_H = \frac{1}{4\alpha_2 r_H\dot{\mathcal{G}}_2}\left[r_H^2 \pm \sqrt{r_H^4 - \frac{96\alpha_2^2\dot{\mathcal{G}}_2^2}{\beta}}\right].$$

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- This analysis suggests that there are two different branches in Teleparallel sGB having asymptotically flat scalarized black hole configurations.

- By Taking $\beta = 4$ (kinetic constant), and setting the background value of the scalar field to zero, we have two coupling constants α_2 and α_3 and two coupling functions $\mathcal{G}_2$ and $\mathcal{G}_3$.

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- We have decided to fix the two coupling functions in the following form

$$\mathcal{G}_2(\psi) = \frac{1}{12}\left(1 - e^{-6\psi^2}\right) = \mathcal{G}_3(\psi)\,.$$

This exponential function has one of the desired properties for scalarization, namely, it allows the GR solutions with a zero scalar field to be also solutions of the more general system of equations in Teleparallel gravity.

- By Taking $\beta = 4$ (kinetic constant), and setting the background value of the scalar field to zero, we have two coupling constants α_2 and α_3 and two coupling functions $\mathcal{G}_2$ and $\mathcal{G}_3$.
- We have decided to fix the two coupling functions in the following form

$$\mathcal{G}_2(\psi) = \frac{1}{12}\left(1 - e^{-6\psi^2}\right) = \mathcal{G}_3(\psi)\,.$$

  This exponential function has one of the desired properties for scalarization, namely, it allows the GR solutions with a zero scalar field to be also solutions of the more general system of equations in Teleparallel gravity.
- For this coupling, it was proven in D. D. Doneva and S. S. Yazadjiev, Phys. Rev. Lett. **120** (2018) no.13, 131103 that stable scalarized black hole solutions exist in the sGB case.

- Having fixed $\mathcal{G}_2$ and $\mathcal{G}_3$, the only theory parameters left to vary are α_2 and α_3 and more precisely, their relative weight. The following two cases are especially interesting, since they are purely Teleparallel, i.e. *the scalarization is triggered by torsion*:

[3] D. D. Doneva, F. M. Ramazanoğlu, H. O. Silva, T. P. Sotiriou and S. S. Yazadjiev, [arXiv:2211.01766 [gr-qc]].

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  In either of these two limiting cases, no contribution of the Riemannian Gauss-Bonnet term is present.
- These cases go beyond the classification of theories allowing for scalarization that is discussed in a recent Review[3]

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- First, In order to gain some intuition about the existence and behavior of black hole solutions let us start with discussing the bifurcation point, which corresponds to the point where Schwarzschild becomes unstable and new scalarized solutions originate, as well as the behavior of the scalarized black hole branches.

Numerical solutions - Mass and scalar charge case 1

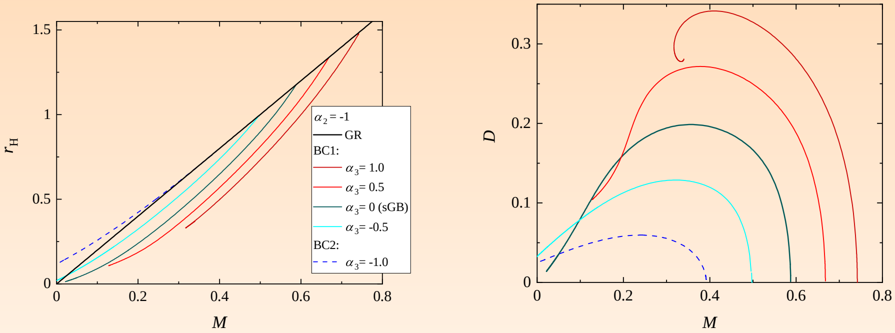

Figure: Setting $\alpha_2 = -1$ (the Riemannian sGB) and varying α_3

- With the increase of α_3 the point of bifurcation from the GR branch moves to large masses.

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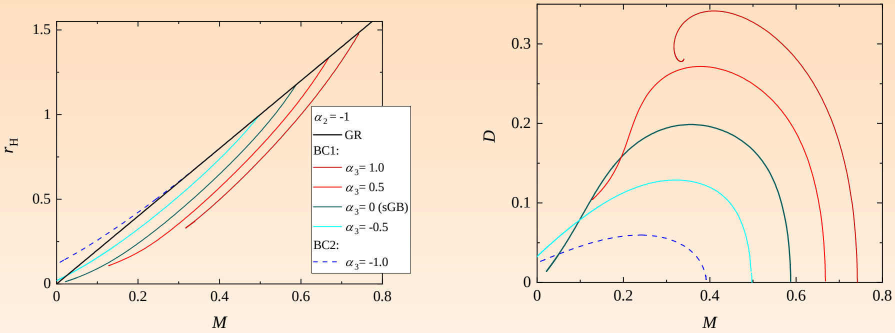


Figure: Setting $\alpha_2 = -1$ (the Riemannian sGB) and varying α_3

- With the increase of α_3 the point of bifurcation from the GR branch moves to large masses.
- For larger α_3, the branch of scalarized solutions disappears at smaller masses.

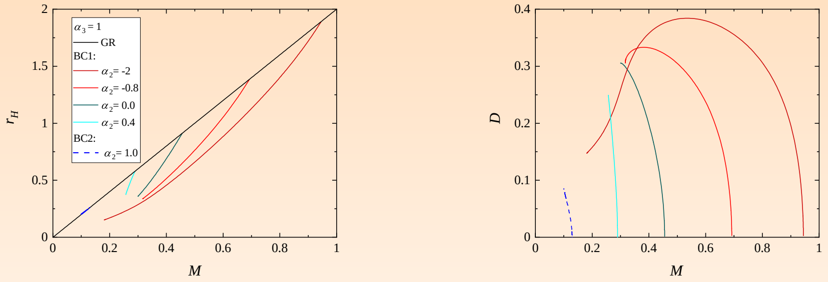
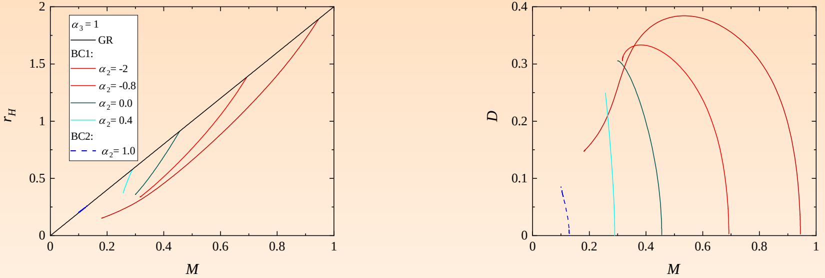


Figure: Setting $\alpha_3 = 1$ (the Teleparallel part) and varying α_2

- Contrary to the previous figure, larger α_2 move the bifurcation point to smaller masses

Numerical solutions - Mass and scalar charge case 2



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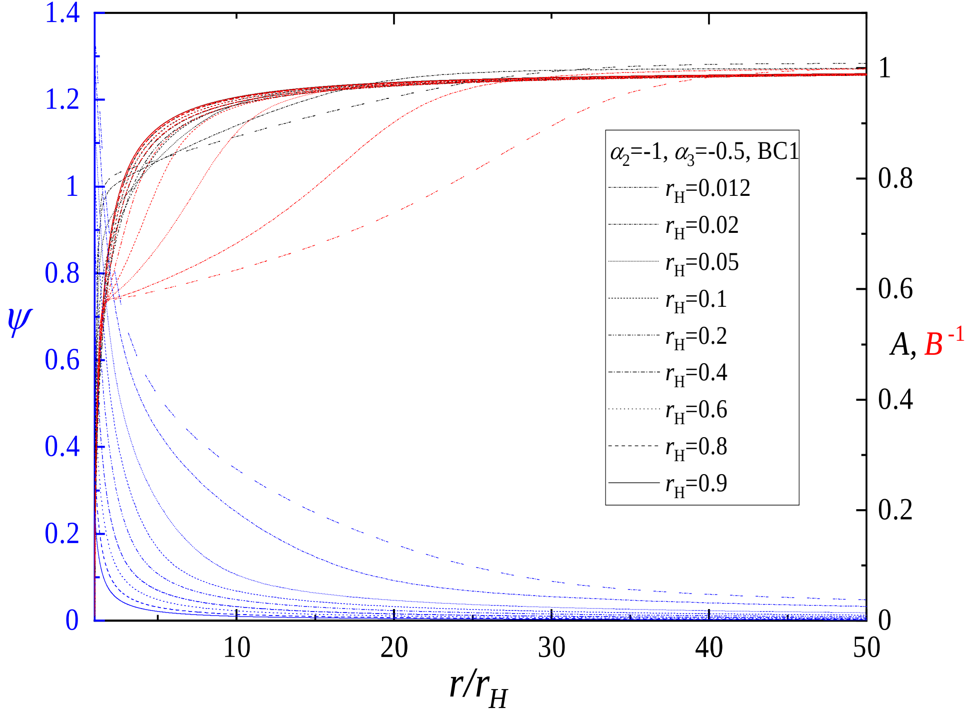
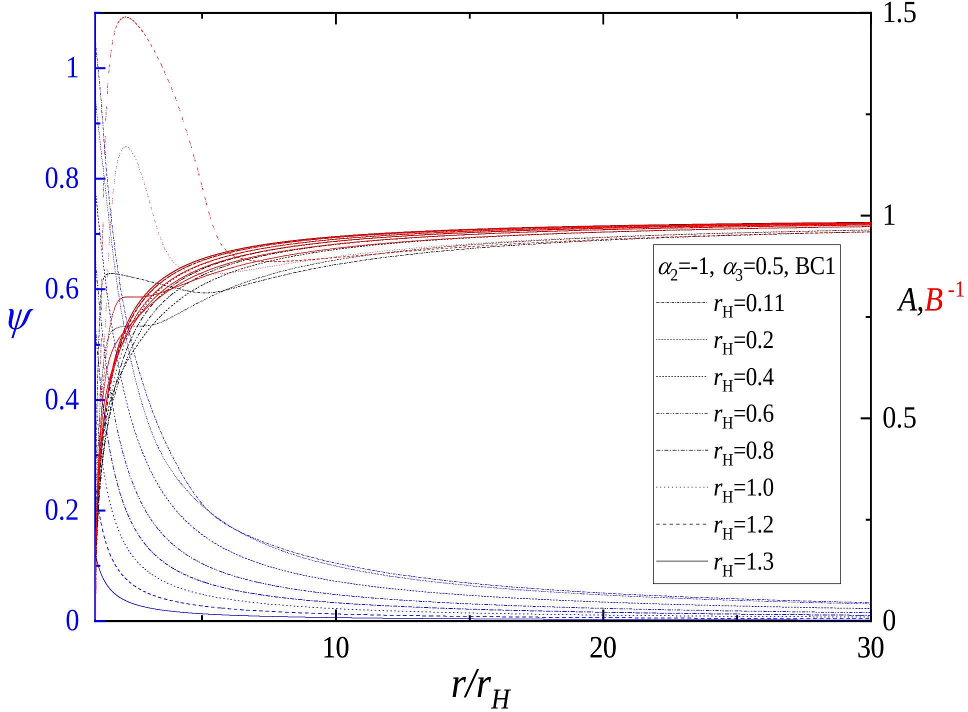
- Contrary to the previous figure, larger α_2 move the bifurcation point to smaller masses
- Even though this case offers a completely new type of scalarization, the behaviour of the solutions branches is qualitatively very similar to the sGB theory



1.4
1.2
1
0.8
0.6
0.4
0.2
0
ψ
10
20
30
40
50
r/r_H
1
0.8
0.6
0.4
0.2
0
A, B^-1
α_2=-1, α_3=-0.5, BC1
r_H=0.012
r_H=0.02
r_H=0.05
r_H=0.1
r_H=0.2
r_H=0.4
r_H=0.6
r_H=0.8
r_H=0.9



ψ
A, B^{-1}
r/r_H
α_2=-1, α_3=0.5, BC1
r_H=0.11
r_H=0.2
r_H=0.4
r_H=0.6
r_H=0.8
r_H=1.0
r_H=1.2
r_H=1.3
0
0.2
0.4
0.6
0.8
1
10
20
30
0.5
1.5

Conclusions and final remarks

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- Interestingly, our theory contains the sGB but there are more going on here.
- In our theory, it is possible to have non-monotonic behavior of the scalar field close to the horizon even for the fundamental nodeless scalar field branch of the black hole. Until now similar non-monotonicity in sGB gravity was observed only for rotating black holes.

Conclusions and final remarks

- Even though we restricted ourselves to static spherically symmetric solutions, the black holes share a lot of similarities not only with the static sGB case but also with the rotating sGB black holes.

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- This observation leads to the question if torsion (or other properties of non-Riemannian geometry) might fundamentally be the origin of the scalarization properties.
- Our study can open a new unexplored window in the study of scalarized black hole solutions in non-Riemannian theories of gravity