Teleparallel Gravity: From Theory to Cosmology

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Outline

Overview of the Talk

1. Introduction to Metric-affine gravity
   - Why modified gravity?
   - Basic ingredients
   - Trinity of gravity
2. Modified Torsional Teleparallel theories of gravity
   - General features
   - Some important theories
   - Applications to cosmology
3. Conclusions and final remarks

General Relativity - Assumptions

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- **Locality**

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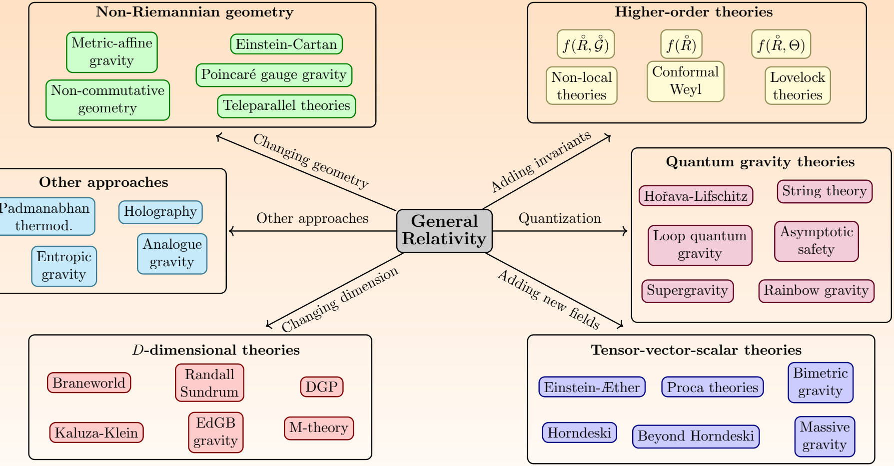
- What is really the inflaton?
- Strong gravity regime needs to be tested;
- A good way to understand GR is to modify it;

How to modify it?



ure: Classification of theories of gravity. (S. Bahamonde et.al., "Teleparallel Gravity: From
ory to Cosmology," 2023, Rep. Prog. Phys. 86 026901.)

Main contributions in cosmology and modified gravity

- **A comprehensive review in dynamical systems in cosmology:** We wrote a very general and important review paper that was published in Physics Report, which explains and summarises all the most important features and results in dynamical systems applied to cosmology. This paper became very popular in the literature as a guide tool to use this powerful technique to understand the cosmology of any theory of gravity.

paper1.png

Fundamental variables and characteristic tensors

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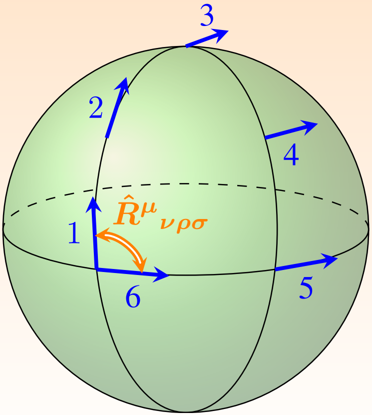
- A priori, there is no relation between $g_{\mu\nu}$ and $\tilde{\Gamma}^{\rho}{}_{\mu\nu}$.
- The most general metric-affine theory is characterised by the following tensors:

Curvature	$\tilde{R}^{\mu}{}_{\nu\rho\sigma} = \partial_{\rho}\tilde{\Gamma}^{\mu}{}_{\nu\sigma} - \partial_{\sigma}\tilde{\Gamma}^{\mu}{}_{\nu\rho} + \tilde{\Gamma}^{\mu}{}_{\tau\rho}\tilde{\Gamma}^{\tau}{}_{\nu\sigma} - \tilde{\Gamma}^{\mu}{}_{\tau\sigma}\tilde{\Gamma}^{\tau}{}_{\nu\rho}$
Torsion	$\tilde{T}^{\mu}{}_{\nu\rho} = \tilde{\Gamma}^{\mu}{}_{\rho\nu} - \tilde{\Gamma}^{\mu}{}_{\nu\rho}$
Non-metricity	$\tilde{Q}_{\mu\nu\rho} = \tilde{\nabla}_{\mu}g_{\nu\rho} = \partial_{\mu}g_{\nu\rho} - \tilde{\Gamma}^{\sigma}{}_{\nu\mu}g_{\sigma\rho} - \tilde{\Gamma}^{\sigma}{}_{\rho\mu}g_{\nu\sigma}$

What does curvature geometrically represent?

Curvature tensor $\tilde{R}^{\alpha}{}_{\beta\mu\nu}$

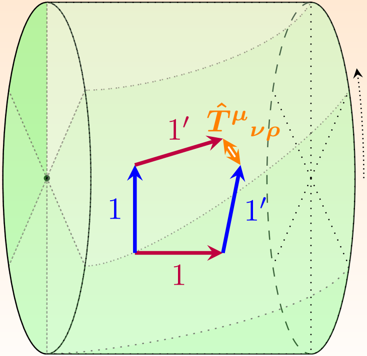
Rotation experienced by a vector when it is parallel transported along a closed curve



What does torsion geometrically represent?

Torsion tensor $\tilde{T}^{\alpha}{}_{\mu\nu}$

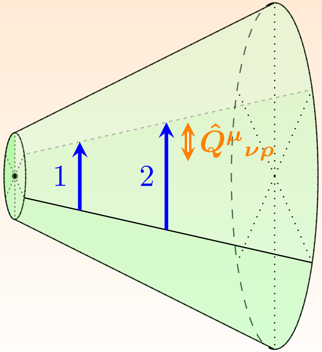
non-closure of the parallelogram formed when two infinitesimal vectors are parallel transported along each other.



What does non-metricity geometrically represent?

Non-metricity tensor $\tilde{Q}_{\alpha\mu\nu}$

measures how much the length and angle of vectors change as we parallel transport them, so in metric spaces the length of vectors is conserve



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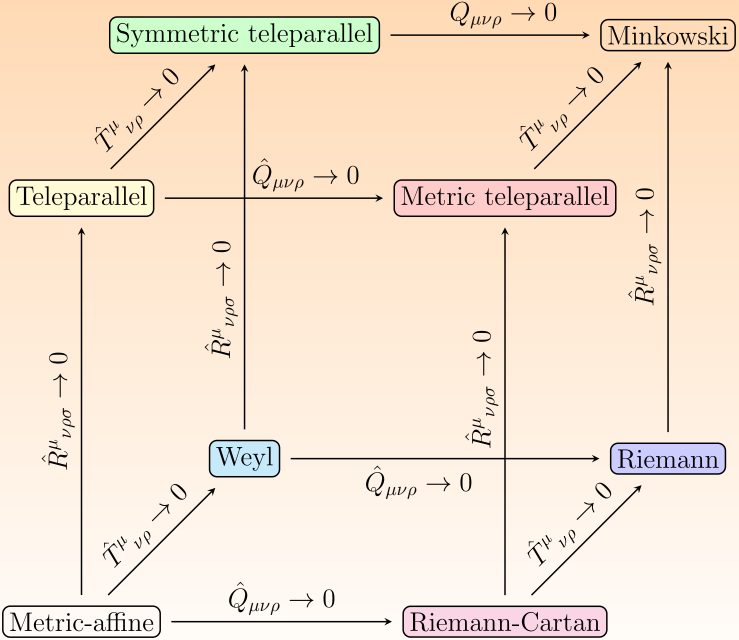

Figure: Classification of metric-affine geometries - Cube

Tetrads and spin connection

- **Notation:** $\mu, \nu, \alpha, ..$: space-time; $a, b, c, ..$: tangent space.
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where η_{ab} is the Minkowski metric.

- Quantities denoted with a circle on top $\circ$ denote that they are defined with respect to the Levi-Civita connection and hats are general affine connection.

Tetrads and spin connection

- The frame coefficients $E_A{}^\mu$ are also required in order to calculate the coefficients $\tilde{\Gamma}^\mu{}_{\nu\rho}$ of the affine connection from the spin connection $\tilde{\omega}^A{}_{B\mu}$ via

$$\tilde{\Gamma}^\rho{}_{\mu\nu} = E_A{}^\rho \left(\partial_\nu e^A{}_\mu + \tilde{\omega}^A{}_{B\nu} e^B{}_\mu\right) ,$$

which is the unique affine connection satisfying the so-called "tetrad postulate"

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- In metric-affine, the spin connection and tetrads are independent to each other.
- It is equivalent to work with either the pair $(g_{\mu\nu}, \Gamma^\mu{}_{\alpha\beta})$ or $(e^A{}_\mu, \omega^A{}_{B\mu})$.

Trinity of gravity - curvature tensor

- The curvature becomes

$$\tilde{R}^{\mu}{}_{\nu\rho\sigma} = \mathring{R}^{\mu}{}_{\nu\rho\sigma} + \mathring{\nabla}_{\rho}\tilde{D}^{\mu}{}_{\nu\sigma} - \mathring{\nabla}_{\sigma}\tilde{D}^{\mu}{}_{\nu\rho} + \tilde{D}^{\mu}{}_{\tau\rho}\tilde{D}^{\tau}{}_{\nu\sigma} - \tilde{D}^{\mu}{}_{\tau\sigma}\tilde{D}^{\tau}{}_{\nu\rho}\,.$$

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Ricci scalar decomposition

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with

$$T := T^{\rho\lambda\kappa}T_{\rho\lambda\kappa} + 2T^{\rho\lambda\kappa}T_{\kappa\rho\lambda} - 4T_{\rho}{}^{\kappa}{}_{\kappa}T^{\rho\lambda}{}_{\lambda}\,, \quad \text{Torsion scalar}\,,$$

$$Q := -\frac{1}{4}Q_{\alpha\beta\gamma}Q^{\alpha\beta\gamma} + \frac{1}{2}Q_{\alpha\beta\gamma}Q^{\beta\alpha\gamma} + \frac{1}{4}Q_{\alpha}Q^{\alpha} - \frac{1}{2}Q_{\alpha}\bar{Q}^{\alpha}\,, \ \text{Nonmetricity scalar}\,,$$

$$C := 2(Q_{\kappa\rho\lambda}T^{\lambda\kappa\rho} + Q_{\rho}{}^{\sigma}{}_{\sigma}T^{\rho\kappa}{}_{\kappa} - Q^{\sigma}{}_{\sigma\rho}T^{\rho\kappa}{}_{\kappa})\,.$$

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Einstein-Hilbert action

$$S_{\rm GR} = \int \left[-\frac{1}{2\kappa^2}\mathring{R} + L_{\rm m}\right]\sqrt{-g}\, d^4x\,.$$

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$$S_{\rm GR} = \int \left[-\frac{1}{2\kappa^2}\mathring{R} + L_{\rm m}\right]\sqrt{-g}\, d^4x\,.$$

Trinity of gravity - General Relativity

- GR assumes **zero torsion and nonmetricity** so that

Ricci scalar GR

$$\tilde{R} = \mathring{R} + \cancel{\left(T - 2\mathring{\nabla}_\mu T^\rho{}_\rho{}^\mu\right)} + \cancel{\left(Q + \mathring{\nabla}_\mu Q^{\mu\nu}{}_\nu - \mathring{\nabla}_\nu Q_\mu{}^{\mu\nu}\right)} + \cancel{C} = \mathring{R}\,.$$

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where $\kappa^2 = 8\pi G$ and $L_{\rm m}$ is any matter Lagrangian.

- The Einstein's field equations are obtained by taking variations w/r to the metric: $\boxed{\mathring{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathring{R} = \kappa^2 T_{\mu\nu}}$.

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Trinity of gravity - Torsional Teleparallel equivalent of GR

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- Let's recall the curvature equation and assume the Teleparallel condition:

$$R^{\sigma}{}_{\rho\mu\nu} = \mathring{R}^{\sigma}{}_{\rho\mu\nu} - \mathring{\nabla}_{\nu} L^{\sigma}{}_{\mu\rho} + \mathring{\nabla}_{\mu} L^{\sigma}{}_{\nu\rho} - L^{\sigma}{}_{\nu\lambda} L^{\lambda}{}_{\mu\rho} + L^{\sigma}{}_{\mu\lambda} L^{\lambda}{}_{\nu\rho} = 0\,,$$

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- By further assuming that non-metricity is zero, then Λ_{μ} coincides with a local Lorentz matrix. Then, the spin connection is pure gauge.
- In this case it is always possible to choose a gauge (at least locally, on a simply connected domain) such that the spin connection vanishes identically, $w^{a}{}_{b\mu} = 0$. This gauge is known as the Weitzenböck gauge. By choosing that gauge, Local Lorentz is broken.

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- Thus, the maximum number of dof in the nonmetricity part of the connection goes down from 40 dof to a maximum of 4 independent dof (that can be expressed via the vector ξ^{μ}).

What happens to the connection in Symmetric Teleparallel gravity?

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- Thus, the maximum number of dof in the nonmetricity part of the connection goes down from 40 dof to a maximum of 4 independent dof (that can be expressed via the vector ξ^{μ}).
- Note that it is possible to choose a gauge such that the (affine)connection is always vanishing (Coincident gauge) - $\Gamma^{\alpha}{}_{\mu\nu} = 0$. By choosing that gauge, diffeo is broken (locally).

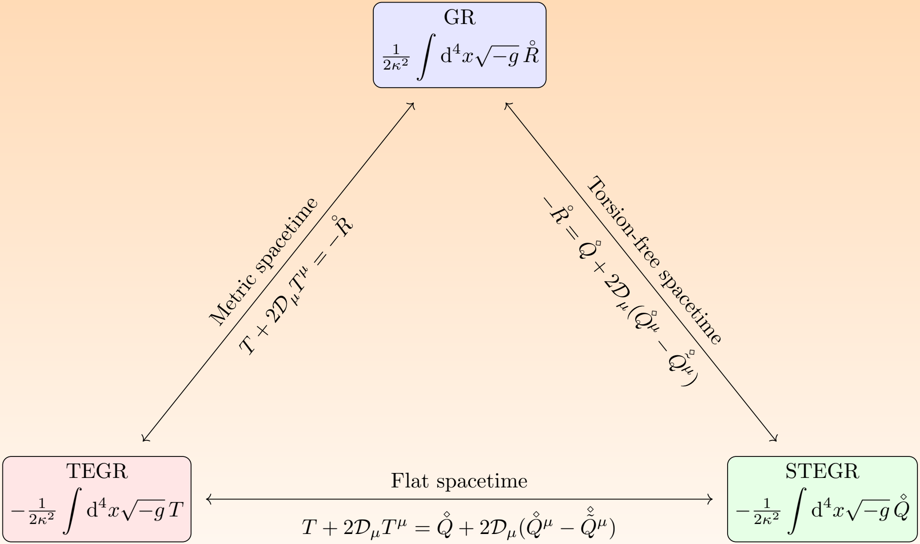

Figure: Geometrical trinity of gravity (S. Bahamonde et.al., "Teleparallel Gravity: From Theory o Cosmology," Rep. Prog. Phys. 86 026901 (2023).; J. Beltrán Jiménez, L. Heisenberg and T. S. Koivisto, "The Geometrical Trinity of Gravity," Universe **5** (2019) no.7, 173.)

Three different ways of understanding gravity

Coupling to matter

In TG, no direct matter coupling to the teleparallel connections are introduced, in order to preserve the weak equivalence principle $\Longrightarrow$ matter fields retain their universal **coupling to the metric** and possibly its **Levi-Civita connection** (in the case of spinor fields).

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Equivalence on their field equations

TEGR and STEGR have the same equations as GR, so **CLASSICALLY** it is impossible to make any observation to distinguish between them. All **classical experiments** already done, that have confirmed GR, also can be understood as a confirmation of TEGR and STEGR.

Overview of the Talk

Modified teleparallel theories

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If we modify the TEGR action, a priori there is no equivalence between modified theories from GR and modified Teleparallel theories.

Are Torsional Teleparallel theories Local Lorentz invariant?

By choosing the Weitzenbock gauge $\omega^{a}{}_{b\mu}=0$, local Lorentz is broken. However, by not adopting any gauge, these theories are invariant under a simultaneous local Lorentz transformations in both the tetrad and spin connection.

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- The field strength in Torsional TG is the torsion tensor that is defined as the antisymmetric part of the Weitzenböck connection

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$$T^{\rho}{}_{\mu\nu} = \Gamma^{\rho}{}_{\nu\mu} - \Gamma^{\rho}{}_{\mu\nu} = E_A{}^{\rho}\Big(e^{A}{}_{\nu,\mu} - e^{A}{}_{\mu,\nu} + \omega^{A}{}_{B\mu}e^{B}{}_{\nu} - \omega^{A}{}_{B\nu}e^{B}{}_{\mu}\Big)\,.$$

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- The torsion tensor is generally non-vanishing, and transforms covariantly under both diffeomorphisms and local Lorentz transformations.
- The pure tetrad formalism was the initial framework used for TG, which chooses a specific frame where the spin connection $\omega^{a}{}_{b\mu}$ vanishes. Be careful choosing the correct tetrad which is compatible with this gauge.

- Variations of any action should be taken with respect to both variables (with the emphasis that the spin connection is non-arbitrary but always flat).

Important properties of Torsional Teleparallel theories

- Variations of any action should be taken with respect to both variables (with the emphasis that the spin connection is non-arbitrary but always flat).
- Since $\omega^{A}{}_{B\mu}$ is a pure-gauge quantity, it can be shown that the the antisymmetric part of the field equations arising from variations w/r to the tetrads $e^{A}{}_{\mu}$ coincides with the variations of the action w/r to $\omega^{A}{}_{B\mu}$.

New General Relativity (NGR)

The torsion tensor can be decomposed in its irreducible parts as

$$a_\mu = \frac{1}{6}\epsilon_{\mu\nu\sigma\rho}T^{\nu\sigma\rho}\,, \quad v_\mu = T^\sigma{}_{\sigma\mu}\,,$$
$$t_{\sigma\mu\nu} = \frac{1}{2}\left(T_{\sigma\mu\nu} + T_{\mu\sigma\nu}\right) + \frac{1}{6}\left(g_{\nu\sigma}v_\mu + g_{\nu\mu}v_\sigma\right) - \frac{1}{3}g_{\sigma\mu}v_\nu\,,$$

where $\epsilon_{\mu\nu\sigma\rho}$ is the totally anti-symmetric Levi-Civita symbol.
From these we build the scalars

$$T_{\text{ax}} = a_\mu a^\mu\,, \quad T_{\text{vec}} = v_\mu v^\mu\,, \quad T_{\text{ten}} = t_{\sigma\mu\nu}t^{\sigma\mu\nu}\,,$$

and the torsion scalar is a linear combination

$$T = \frac{3}{2}T_{\text{ax}} + \frac{2}{3}T_{\text{ten}} - \frac{2}{3}T_{\text{vec}}\,.$$

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- If $c_1 = -\frac{2}{3}$, $c_2 = \frac{3}{2}$, $c_3 = \frac{2}{3}$, the above action is equivalent to the TEGR one.

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- This theory contains parity-preserving quadratic form of the torsion with three free parameters.
- Perturbations around Minkowski shows that the unique stable Minkowski background that includes gravity is the TEGR case[2].

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- **Strongly coupling problem?** By performing Minkowski perturbations, one only finds new modes at 4th order in the perturbation (J. Beltrán Jiménez, A. Golovnev, T. Koivisto and H. Veermäe, [arXiv:2004.07536])

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- It is possible to extend this theory by adding more invariants. One interesting theory is when one considers[4]

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$$\mathcal{S}_{f(T,B)} = \int f(T,B) e\, d^4x\,.$$

- If $f(T,B) = f(-T+B) = f(\mathring{R})$, one finds the $f(\mathring{R})$ theory in the context of TEGR.
- If $f(T,B) = f(T)$, one gets $f(T)$ gravity
- Other theories related to the boundary term such as $-T + f(B)$ gravity.

[4] S. Bahamonde, C. G. Böhmer and M. Wright, Phys. Rev. D **92** (2015) no.10, 104042

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- The speed of the gravitational waves is exactly c and extra polarization models, namely the longitudinal and breathing modes, do appear at first-order perturbation level.

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FLRW tetrad compatible with cosmological symmetries in the Weitzenböck gauge

$$e^a{}_\mu = \mathrm{diag}(N(t), a(t), a(t), a(t))$$
$$\to ds^2 = N(t)^2 - a(t)^2(dx^2 + dy^2 + dz^2)\,.$$

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Antisymmetric field equations

Important point: the tetrad showed in the last slide in the Weitzenböck gauge solves all the antisymmetric field equations for any Teleparallel gravitational theory

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Spherical coordinates

Be careful here: In spherical coordinates (t, r, θ, ϕ), the tetrad in the Weitzenböck gauge looks more complicated (off-diagonal terms appear):

$$e^a{}_\mu = \begin{pmatrix} N(t) & 0 & 0 & 0 \\ 0 & a(t)\sin(\theta)\cos(\phi) & ra(t)\cos(\theta)\cos(\phi) & -ra(t)\sin(\theta)\sin(\phi) \\ 0 & a(t)\sin(\theta)\sin(\phi) & ra(t)\cos(\theta)\sin(\phi) & ra(t)\sin(\theta)\cos(\phi) \\ 0 & a(t)\cos(\theta) & -ra(t)\sin(\theta) & 0 \end{pmatrix}.$$

Cosmological perturbations in TG

- In a metrical theory, one perturbs the FLRW metric in the scalar-vector-tensor decomposition form:

$$\delta g_{\mu\nu} = \begin{bmatrix} -2\varphi & a\left(\partial_i \mathcal{B} + \mathcal{B}_i\right) \\ a\left(\partial_i \mathcal{B} + \mathcal{B}_i\right) & 2a^2\left(-\psi\delta_{ij} + \partial_i\partial_j h + 2\partial_{(i}h_{j)} + \frac{1}{2}h_{ij}\right) \end{bmatrix}.$$

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- The **metric has 10 d.o.f.** (4 scalars(1 each), 2 vectors(2 each), 1 tensor(2 each)) and the **tetrads 16 d.o.f.** (6 scalars(1 each), 4 vectors(2 each), 1 tensor(2 each)).

Background cosmology in $f(T,B)$ gravity

- The modified FLRW equations in $f(T,B)$ gravity are[5]

$$\begin{aligned} -3H^2(3f_B+2f_T)+3H\dot{f}_B-3\dot{H}f_B+\frac{1}{2}f &= \kappa^2\rho_m\,, \\ -\left(3H^2+\dot{H}\right)(3f_B+2f_T)-2H\dot{f}_T+\ddot{f}_B+\frac{1}{2}f &= -\kappa^2 p_m\,. \end{aligned}$$

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- It can describe different bounce cosmological solutions.
- I can alleviate both σ_8 and H_0 at the same time.

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Cosmological perturbations in $f(T,B)$ gravity

- **Tensorial perturbations:** GW propagation equation is[6]

$$\ddot{h}_{ij} + (3+\alpha_M)H\dot{h}_{ij} + \frac{k^2}{a^2}h_{ij} = 0\,,$$

meaning that $c_T^2 = 1$ with a Planck mass run rate $\alpha_M = \frac{1}{H}\frac{\dot{f}_T}{f_T}$. Thus, $f_T < 0$ is required for stability issues.

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- **Vectorial perturbations:** The vector perturbations are not propagating (as in $f(\overset{\circ}{R})$).

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- **Scalar perturbations:** Very messy symmetric and antisymmetric field equations. The density parameter and the weak lensing parameter in Fourier space of the sub-horizon limit obey

$$\ddot{\delta}_m + 2H\dot{\delta}_m \simeq -\frac{k^2\varphi}{a^2} = 4\pi\rho G_{\rm eff}\delta_{\rm m} = \frac{\kappa^2}{2}\rho G_{\rm eff}\delta_{\rm m}\,,$$

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- For example for $f_{BB} + 2f_{TB} + f_{TT} = 0$ one finds $G_{\rm eff} = -G\frac{4}{3(f_T+12H^2 f_{TB})}$. One can use these results to constrain models.

- $f(T)$ gravity model does not show tension on the H_0 that prevails in the ΛCDM cosmology, however, σ_8 tension persists(R. C. Nunes, JCAP **05** (2018), 052)

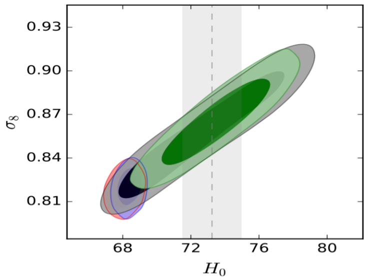

Figure 4. Parametric space in the plane H_0 - σ_8, where the regions in red (blue) show the constraints for ΛCDM model from CMB + BAO (CMB + BAO + H_0), respectively. The regions in black (green) show the constraints for $f(T)$ gravity from CMB + BAO (CMB + BAO + H_0), respectively. The vertical gray band corresponds to $H_0 = 73.24 \pm 1.74$ km s^{-1} Mpc^{-1}.

Background cosmology in Teleparallel scalar-tensor

- Teleparallel dark energy[7] (coupling like $\xi\phi^2 T$) exhibits a quintessence-like, dark-energy-dominated solution, or to the stiff dark-energy late-time attractor, similarly to standard quintessence. There is an additional late-time solution, in which dark energy behaves like a cosmological constant.

[7] C. Q. Geng, C. C. Lee, E. N. Saridakis and Y. P. Wu, Phys. Lett. B **704** (2011), 384-387

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- Theories with a coupling $\chi\phi^2 B$ have late time accelerating attractor solution without requiring any fine tuning of the parameters. A dynamical crossing of the phantom barrier is also possible[8]
- TG non-local cosmology with a term like $Tf(\mathring{\Box}^{-1}T)$ in the action is consistent with the present cosmological data obtained by SNe Ia + BAO + CC + H0 observations[9]

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Teleparallel Horndeski gravity - perturbations

- By considering tensorial perturbations only and after some cumbersome calculations, one gets the following wave equation

$$\ddot{h}_{ij} + (3 + \alpha_{\mathsf{M}})\, H\dot{h}_{ij} - (1 + \alpha_{\mathsf{T}})\, \frac{k^2}{a^2} h_{ij} = 0,$$

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where $\alpha_T = c_T^2 - 1$ and the speed of GW being equal to[10]

Speed of GW in Teleparallel Horndeski

$$c_T^2 = \frac{G_4 - X(\ddot{\phi}G_{5,X} + G_{5,\phi}) - G_{\rm Tele,T}}{G_4 - 2XG_{4,X} - X(H\dot{\phi}G_{5,X} - G_{5,\phi}) + 2XG_{\rm Tele,J_8} + \frac{1}{2}XG_{\rm Tele,J_5} - G_{\rm Tele,T}}\,.$$

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Reviving Horndeski using Teleparallel gravity

- For $G_{\rm Tele}=0$ (standard case), one gets that to achieve a theory consistent with the GW observations $c_T=1$, one requires $G_5(\phi,X)=\text{constant}$ and $G_4(\phi,X)=G_4(\phi)$. Hence, Horndeski gravity is highly constraint.

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Teleparallel Lagrangian respecting $c_T = 1$ ($\alpha_T = 0$)

$$\mathcal{L} = \tilde{G}_{\text{tele}}(\phi, X, T, T_{\text{vec}}, I_2) + \sum_{i=2}^{4} \mathcal{L}_i + G_5(\phi)\mathcal{G}_{\mu\nu}\phi^{;\mu\nu} \,.$$

Overview of the Talk

Conclusions

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- It is possible to formulate theories which are equivalent to GR, and then, one can modify these equations to explain dark energy or inflation.
- One needs to be more careful than in Riemannian theories since the tetrad and spin connection form a pair that always need to be considered in a proper way to fulfill the symmetry condition to then solve the antisymmetric field equations.

Conclusions

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- There are many things totally unexplored in TG, so please go ahead!
- I did not have time to explain our recent paper 2212.08005 where we formulated Symmetric Teleparallel Horndeski gravity (only with non-metricity).