Black holes solutions in metric-affine gravity with dynamical torsion and nonmetricity

Sebastián Bahamonde

Postdoctoral Researcher at Tokyo Institute of Technology, Japan

Recent Advances in Theoretical Cosmology and Astrophysics

Based on JCAP **09** (2020), 057, Eur.Phys.J.C **81** (2021) 6, 495;
arXiv:2108.12414 (to appear in JCAP); Jointly with Jorge Gigante Valcarcel.

TARTU ÜLIKOOL · UNIVERSITAS TARTUENSIS · 1632

東京工業大学 Tokyo Institute of Technology

Outline

1. Brief introduction to Metric-affine gravity
   - Dynamics
2. MAG models with dynamical torsion and nonmetricity
   - Spherical symmetry
   - Observational constraints
   - Axial symmetry

Fundamental variables and characteristic tensors

- In the most general metric-affine setting, the fundamental variables are a **metric** $g_{\mu\nu}$ (10 comp.) as well as the coefficients $\tilde{\Gamma}^{\rho}{}_{\mu\nu}$ (64 comp.) of an **affine connection.**

Fundamental variables and characteristic tensors

- In the most general metric-affine setting, the fundamental variables are a **metric** $g_{\mu\nu}$ (10 comp.) as well as the coefficients $\tilde{\Gamma}^{\rho}{}_{\mu\nu}$ (64 comp.) of an **affine connection.**
- The most general connection can be written as

Fundamental variables and characteristic tensors

- In the most general metric-affine setting, the fundamental variables are a **metric** $g_{\mu\nu}$ (10 comp.) as well as the coefficients $\tilde{\Gamma}^{\rho}{}_{\mu\nu}$ (64 comp.) of an **affine connection.**
- The most general connection can be written as

Connection decomposition

$$\tilde{\Gamma}^{\lambda}{}_{\mu\nu} = \overbrace{\Gamma^{\lambda}{}_{\mu\nu}}^{\textbf{Levi-Civita}}$$

Fundamental variables and characteristic tensors

- In the most general metric-affine setting, the fundamental variables are a **metric** $g_{\mu\nu}$ (10 comp.) as well as the coefficients $\tilde{\Gamma}^{\rho}{}_{\mu\nu}$ (64 comp.) of an **affine connection.**
- The most general connection can be written as

Connection decomposition

$$\tilde{\Gamma}^{\lambda}{}_{\mu\nu} = \overbrace{\Gamma^{\lambda}{}_{\mu\nu}}^{\textbf{Levi-Civita}}$$

Fundamental variables and characteristic tensors

- In the most general metric-affine setting, the fundamental variables are a **metric** $g_{\mu\nu}$ (10 comp.) as well as the coefficients $\tilde{\Gamma}^{\rho}{}_{\mu\nu}$ (64 comp.) of an **affine connection.**
- The most general connection can be written as

Connection decomposition

$$\tilde{\Gamma}^{\lambda}{}_{\mu\nu} = \overbrace{\Gamma^{\lambda}{}_{\mu\nu}}^{\textbf{Levi-Civita}} + \overbrace{\frac{1}{2}T^{\lambda}{}_{\mu\nu} - T_{(\mu}{}^{\lambda}{}_{\nu)}}^{\textbf{Torsion part}}$$

Fundamental variables and characteristic tensors

- In the most general metric-affine setting, the fundamental variables are a **metric** $g_{\mu\nu}$ (10 comp.) as well as the coefficients $\tilde{\Gamma}^{\rho}{}_{\mu\nu}$ (64 comp.) of an **affine connection.**
- The most general connection can be written as

Connection decomposition

$$\tilde{\Gamma}^{\lambda}{}_{\mu\nu} = \overbrace{\Gamma^{\lambda}{}_{\mu\nu}}^{\text{Levi-Civita}} + \overbrace{\frac{1}{2}T^{\lambda}{}_{\mu\nu} - T_{(\mu}{}^{\lambda}{}_{\nu)}}^{\text{Torsion part}} + \overbrace{\frac{1}{2}Q^{\lambda}{}_{\mu\nu} - Q_{(\mu}{}^{\lambda}{}_{\nu)}}^{\text{Nonmetricity part}}, \quad (1)$$

Fundamental variables and characteristic tensors

- In the most general metric-affine setting, the fundamental variables are a **metric** $g_{\mu\nu}$ (10 comp.) as well as the coefficients $\tilde{\Gamma}^{\rho}{}_{\mu\nu}$ (64 comp.) of an **affine connection.**
- The most general connection can be written as

Connection decomposition

$$\tilde{\Gamma}^{\lambda}{}_{\mu\nu} = \overbrace{\Gamma^{\lambda}{}_{\mu\nu}}^{\text{Levi-Civita}} + \overbrace{\frac{1}{2}T^{\lambda}{}_{\mu\nu} - T_{(\mu}{}^{\lambda}{}_{\nu)}}^{\text{Torsion part}} + \overbrace{\frac{1}{2}Q^{\lambda}{}_{\mu\nu} - Q_{(\mu}{}^{\lambda}{}_{\nu)}}^{\text{Nonmetricity part}}, \quad (1)$$

Curvature	$\tilde{R}^{\mu}{}_{\nu\rho\sigma} = \partial_{\rho}\tilde{\Gamma}^{\mu}{}_{\nu\sigma} - \partial_{\sigma}\tilde{\Gamma}^{\mu}{}_{\nu\rho} + \tilde{\Gamma}^{\mu}{}_{\tau\rho}\tilde{\Gamma}^{\tau}{}_{\nu\sigma} - \tilde{\Gamma}^{\mu}{}_{\tau\sigma}\tilde{\Gamma}^{\tau}{}_{\nu\rho}$
Torsion	$\tilde{T}^{\mu}{}_{\nu\rho} = \tilde{\Gamma}^{\mu}{}_{\rho\nu} - \tilde{\Gamma}^{\mu}{}_{\nu\rho}$
Nonmetricity	$\tilde{Q}_{\mu\nu\rho} = \tilde{\nabla}_{\mu}g_{\nu\rho} = \partial_{\mu}g_{\nu\rho} - \tilde{\Gamma}^{\sigma}{}_{\nu\mu}g_{\sigma\rho} - \tilde{\Gamma}^{\sigma}{}_{\rho\mu}g_{\nu\sigma}$

Dynamics of metric-affine geometry

- Gravitational action with dynamical torsion and nonmetricity:

$$S = \int d^4x\sqrt{-g}\left[\mathcal{L}_m - \frac{1}{16\pi}\mathcal{L}_g(\tilde{\mathcal{R}},\mathcal{T},\mathcal{Q})\right]. \qquad (2)$$

Dynamics of metric-affine geometry

- Gravitational action with dynamical torsion and nonmetricity:

$$S = \int d^4x\sqrt{-g}\left[\mathcal{L}_m - \frac{1}{16\pi}\mathcal{L}_g(\tilde{\mathcal{R}}, \mathcal{T}, \mathcal{Q})\right] . \tag{2}$$

- Correspondence between geometry and matter:

$$\frac{\delta S_g}{\delta e^a{}_\nu} = 16\pi\theta_a{}^\nu , \tag{3}$$

$$\frac{\delta S_g}{\delta \omega^a{}_{b\nu}} = 16\pi\Delta_a{}^{b\nu} . \tag{4}$$

Here $\theta_a{}^\nu$ is the energy-momentum tensor (canonical) and $\Delta_a{}^{b\nu}$ is the hypermomentum density tensor.

MAG models with dynamical torsion and nonmetricity

- Quadratic gravitational action with dynamical torsion and nonmetricity in Weyl-Cartan geometry ($Q_{\lambda\mu\nu} = g_{\mu\nu}W_\lambda$)

$$S = \int d^4x\sqrt{-g}\Big\{\mathcal{L}_{\rm m} + \frac{1}{64\pi}\Big[-4R - 6d_1\tilde{R}_{\lambda[\rho\mu\nu]}\tilde{R}^{\lambda[\rho\mu\nu]} - 9d_1\tilde{R}_{\lambda[\rho\mu\nu]}\tilde{R}^{\mu[\lambda\nu\rho]} + 8\,d_1\tilde{R}_{[\mu\nu]}\tilde{R}^{[\mu\nu]} + \frac{1}{8}\left(32e_1 + 8e_2 + 17d_1\right)\tilde{R}^{\lambda}{}_{\lambda\mu\nu}\tilde{R}^{\rho}{}_{\rho}{}^{\mu\nu} - 7d_1\tilde{R}_{[\mu\nu]}\tilde{R}^{\lambda}{}_{\lambda}{}^{\mu\nu} + 3\left(1-2a_2\right)T_{[\lambda\mu\nu]}T^{[\lambda\mu\nu]}\Big]\Big\}\,.$$

[1] S. Bahamonde and J. G. Valcarcel, JCAP **09**, 057 (2020).

[2] S. Bahamonde and J. G. Valcarcel, arXiv: 2108.12414, to be published in JCAP.

MAG models with dynamical torsion and nonmetricity

- Quadratic gravitational action with dynamical torsion and nonmetricity in Weyl-Cartan geometry ($Q_{\lambda\mu\nu} = g_{\mu\nu}W_\lambda$)

$$S = \int d^4x\sqrt{-g}\Big\{\mathcal{L}_{\rm m} + \frac{1}{64\pi}\Big[-4R - 6d_1\tilde{R}_{\lambda[\rho\mu\nu]}\tilde{R}^{\lambda[\rho\mu\nu]} - 9d_1\tilde{R}_{\lambda[\rho\mu\nu]}\tilde{R}^{\mu[\lambda\nu\rho]} + 8\,d_1\tilde{R}_{[\mu\nu]}\tilde{R}^{[\mu\nu]} + \frac{1}{8}\left(32e_1 + 8e_2 + 17d_1\right)\tilde{R}^{\lambda}{}_{\lambda\mu\nu}\tilde{R}^{\rho}{}_{\rho}{}^{\mu\nu} - 7d_1\tilde{R}_{[\mu\nu]}\tilde{R}^{\lambda}{}_{\lambda}{}^{\mu\nu} + 3\left(1-2a_2\right)T_{[\lambda\mu\nu]}T^{[\lambda\mu\nu]}\Big]\Big\}.$$

- Absence of a general Birkhoff's theorem in MAG: new spherically and axially symmetric vacuum solutions with independent dynamical torsion and nonmetricity fields[1,2].

[1] S. Bahamonde and J. G. Valcarcel, JCAP **09**, 057 (2020).
[2] S. Bahamonde and J. G. Valcarcel, arXiv: 2108.12414, to be published in JCAP.

Spherical symmetry

- Metric, torsion and nonmetricity in spherically symmetric space-times ($\#2 + \#8 + \#2 = \#12$):

$$\mathcal{L}_\xi g_{\mu\nu} = \mathcal{L}_\xi T^\lambda{}_{\mu\nu} = \mathcal{L}_\xi W_\mu = 0 \implies \mathcal{L}_\xi \tilde{R}_{\lambda\rho\mu\nu} = 0$$

Spherical symmetry

- Metric, torsion and nonmetricity in spherically symmetric space-times ($\#2+\#8+\#2=\#12$):

$$\mathcal{L}_{\xi} g_{\mu\nu} = \mathcal{L}_{\xi} T^{\lambda}{}_{\mu\nu} = \mathcal{L}_{\xi} W_{\mu} = 0 \implies \mathcal{L}_{\xi} \tilde{R}_{\lambda\rho\mu\nu} = 0$$

- By solving these equations we find that torsion and nonmetricity behave as

$$T^{t}{}_{tr} = a(r)\,, \quad T^{r}{}_{tr} = b(r)\,, \quad T^{\theta_k}{}_{t\theta_k} = f(r)\,, \quad T^{\theta_k}{}_{r\theta_k} = g(r)$$
$$T^{\theta_k}{}_{t\theta_l} = e^{a\theta_k}\, e^{b}{}_{\theta_l}\, \epsilon_{ab}\, d(r)\,, \quad T^{\theta_k}{}_{r\theta_l} = e^{a\theta_k}\, e^{b}{}_{\theta_l}\, \epsilon_{ab}\, h(r)\,,$$
$$T^{t}{}_{\theta_k\theta_l} = \epsilon_{kl}\, k(r)\, \sin\theta_1\,, \quad T^{r}{}_{\theta_k\theta_l} = \epsilon_{kl}\, l(r)\, \sin\theta_1\,,$$
$$W_{\lambda} = (w_1(r), w_2(r), 0, 0)\,,$$

whereas the metric is in the standard spherically symmetric form:

$$ds^2 = \Psi_1(r)\, dt^2 - \frac{dr^2}{\Psi_2(r)} - r^2\left(d\theta_1^2 + \sin^2\theta_1 d\theta_2^2\right)\,.$$

Here, ϵ_{kl} is the Levi-Civita symbol in two dimensions.

Spherical symmetry - Solving the field equations

The field eqs are very involved. To solve them we use the following strategy:

1. **Imposing regularity:** In general, the solutions can have a singular behaviour. To ensure regularity, one can analyse the torsion/nonmetricity tensors referred to the rotated basis $\vartheta^a = \Lambda^a{}_b e^b$.

Spherical symmetry - Solving the field equations

The field eqs are very involved. To solve them we use the following strategy:

1. **Imposing regularity:** In general, the solutions can have a singular behaviour. To ensure regularity, one can analyse the torsion/nonmetricity tensors referred to the rotated basis $\vartheta^a = \Lambda^a{}_b e^b$.

Spherical symmetry - Solving the field equations

The field eqs are very involved. To solve them we use the following strategy:

1. **Imposing regularity:** In general, the solutions can have a singular behaviour. To ensure regularity, one can analyse the torsion/nonmetricity tensors referred to the rotated basis $\vartheta^a = \Lambda^a{}_b e^b$.
   One can write the gauge curvature $\mathcal{F}^a{}_{bc} = \vartheta^a{}_\lambda \vartheta_b{}^\mu \vartheta_c{}^\nu T^\lambda{}_{\nu\mu}$ related to the torsion/nonmetricity tensor in this orthogonal coframe.

Spherical symmetry - Solving the field equations

The field eqs are very involved. To solve them we use the following strategy:

1. **Imposing regularity:** In general, the solutions can have a singular behaviour. To ensure regularity, one can analyse the torsion/nonmetricity tensors referred to the rotated basis $\vartheta^a = \Lambda^a{}_b e^b$.
One can write the gauge curvature $\mathcal{F}^a{}_{bc} = \vartheta^a{}_\lambda \vartheta_b{}^\mu \vartheta_c{}^\nu T^\lambda{}_{\nu\mu}$ related to the torsion/nonmetricity tensor in this orthogonal coframe.
Regularity restricts the initial arbitrariness of the torsion components and the Weyl vector by imposing the relations

$$b(r) = a(r)\sqrt{\Psi_1(r)\Psi_2(r)}\,, \qquad f(r) = -g(r)\sqrt{\Psi_1(r)\Psi_2(r)}\,,$$
$$d(r) = -h(r)\sqrt{\Psi_1(r)\Psi_2(r)}\,, \quad l(r) = k(r)\sqrt{\Psi_1(r)\Psi_2(r)}\,,$$
$$w_1(r) = -w_2(r)\sqrt{\Psi_1(r)\Psi_2(r)}\,.$$

Spherical symmetry - Solving the field equations

2. **Solve the weak field limit:** The weak field limit of the field equations become

$$\nabla_\rho \nabla_\lambda T^{\lambda\rho}{}_\mu + \nabla_\rho \nabla^\rho T^\lambda{}_{\mu\lambda} - \nabla_\rho \nabla_\mu T^{\lambda\rho}{}_\lambda = 0\,,$$
$$\nabla_\mu \tilde{R}^\lambda{}_\lambda{}^{\mu\nu} = 0\,.$$

Spherical symmetry - Solving the field equations

2. **Solve the weak field limit:** The weak field limit of the field equations become

$$\nabla_\rho \nabla_\lambda T^{\lambda\rho}{}_{\mu} + \nabla_\rho \nabla^\rho T^{\lambda}{}_{\mu\lambda} - \nabla_\rho \nabla_\mu T^{\lambda\rho}{}_{\lambda} = 0\,,$$
$$\nabla_\mu \tilde{R}^{\lambda}{}_{\lambda}{}^{\mu\nu} = 0\,.$$

Spherical symmetry - Solving the field equations

2. **Solve the weak field limit:** The weak field limit of the field equations become

$$\nabla_\rho \nabla_\lambda T^{\lambda\rho}{}_\mu + \nabla_\rho \nabla^\rho T^\lambda{}_{\mu\lambda} - \nabla_\rho \nabla_\mu T^{\lambda\rho}{}_\lambda = 0\,,$$
$$\nabla_\mu \tilde{R}^\lambda{}_\lambda{}^{\mu\nu} = 0\,.$$

These equations can be solved, yielding

$$w_1(r) = -\,\kappa_d \int \sqrt{\frac{\Psi_1(r)}{\Psi_2(r)}}\,\frac{dr}{r^2}\,,$$

$$b(r) = r f'(r) + f(r) + \frac{\kappa_d}{2r}\sqrt{\frac{\Psi_1(r)}{\Psi_2(r)}}\,,$$

where κ_d is an integration constant which represents the dilaton charge.

Spherical symmetry - Solving the field equations

1. The final solution for the metric behaves as Reissner-Nordström

$$g_{tt} = -1/g_{rr} \equiv \Psi(r) = 1 - \frac{2m}{r} + \frac{d_1 \kappa_{\rm s}^2 - 4 e_1 \kappa_{\rm d,e}^2}{r^2} \,. \qquad (5)$$

Spherical symmetry - Solving the field equations

1. The final solution for the metric behaves as Reissner-Nordström

$$g_{tt} = -1/g_{rr} \equiv \Psi(r) = 1 - \frac{2m}{r} + \frac{d_1 \kappa_{\rm s}^2 - 4 e_1 \kappa_{\rm d,e}^2}{r^2} \,. \tag{5}$$

2. Nonmetricity sector:

$$W_\mu = \frac{\kappa_{\rm d,e}}{r} \left(1, -1/\Psi(r), 0, 0\right) \,. \tag{6}$$

Spherical symmetry - Solving the field equations

1. The final solution for the metric behaves as Reissner-Nordström

$$g_{tt} = -1/g_{rr} \equiv \Psi(r) = 1 - \frac{2m}{r} + \frac{d_1\kappa_{\rm s}^2 - 4e_1\kappa_{\rm d,e}^2}{r^2}\,. \tag{5}$$

2. Nonmetricity sector:

$$W_\mu = \frac{\kappa_{\rm d,e}}{r}\left(1, -1/\Psi(r), 0, 0\right)\,. \tag{6}$$

3. Torsion sector:

$$\bar{\mathcal{S}}^a = -\frac{\kappa_{\rm s}}{r}(1,1,0,0)\,, \tag{7}$$

$$\bar{\mathcal{T}}_2^{abc} = \frac{\kappa_{\rm s}}{3r}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}\,. \tag{8}$$

Dilation and spin charges

What do κ_s (dilation charge) and $\kappa_{d,e}$ (spin charge) physically represent?

Point 1 - Hypermomentum density

In the geometric scheme of MAG, not only an energy-momentum tensor of matter arises as source of curvature, but also a hypermomentum density tensor which operates as source of torsion and nonmetricity.

Dilation and spin charges

What do κ_s (dilation charge) and $\kappa_{d,e}$ (spin charge) physically represent?

Point 1 - Hypermomentum density

In the geometric scheme of MAG, not only an energy-momentum tensor of matter arises as source of curvature, but also a hypermomentum density tensor which operates as source of torsion and nonmetricity.

Point 2 - Dilation and spin charges

In Weyl-Cartan geometry, hypermomentum density tensor splits into spin and dilation currents, which carry their own charges and provide a RN solution.

Dilation and spin charges

When these charges might be important?

Significant effects are contemplated only around **extreme gravitational systems**, such as **neutron stars** with intense magnetic fields and sufficiently oriented elementary spins or **black holes** endowed with spin and dilation charges.

Observational constrains

- Let us now consider the case where the effect of torsion dominates over the contribution of nonmetricity.

[3] S. Bahamonde and J. Gigante Valcarcel, Eur. Phys. J. C **81** (2021) no.6, 495.

Observational constrains

- Let us now consider the case where the effect of torsion dominates over the contribution of nonmetricity.
- **Perihelion shift+ Gravitational redshift:** Assuming the same approximation in Sgr A* and considering the universality of the coupling constant d_1, we find[3]

$$1.396 \cdot 10^{10} \leq \frac{\kappa_{s,\mathrm{SgrA}*}}{\kappa_{s,\mathrm{SiriusB}}} \leq 1.688 \cdot 10^{10} \,.$$

[3] S. Bahamonde and J. Gigante Valcarcel, Eur. Phys. J. C **81** (2021) no.6, 495.

Observational constrains

- Let us now consider the case where the effect of torsion dominates over the contribution of nonmetricity.
- **Perihelion shift+ Gravitational redshift:** Assuming the same approximation in Sgr A* and considering the universality of the coupling constant d_1, we find[3]

$$1.396 \cdot 10^{10} \leq \frac{\kappa_{s,\mathrm{SgrA*}}}{\kappa_{s,\mathrm{SiriusB}}} \leq 1.688 \cdot 10^{10} \,.$$

- To the best of our knowledge, this bound provides the first observational comparison between the spin charges of a supermassive black hole and a degenerate star.

[3] S. Bahamonde and J. Gigante Valcarcel, Eur. Phys. J. C **81** (2021) no.6, 495.

Extension to axisymmetric space-times

- Metric, torsion and nonmetricity tensors in symmetric space-times:

$$\mathcal{L}_\xi g_{\mu\nu} = \mathcal{L}_\xi T^\lambda{}_{\mu\nu} = \mathcal{L}_\xi Q^\lambda{}_{\mu\nu} = 0 \implies \mathcal{L}_\xi \tilde{R}^\lambda{}_{\rho\mu\nu} = 0\,. \qquad (9)$$

[4] S. Bahamonde and J. G. Valcarcel, arXiv: 2108.12414, to be published in JCAP.

Extension to axisymmetric space-times

- Metric, torsion and nonmetricity tensors in symmetric space-times:

$$\mathcal{L}_\xi g_{\mu\nu} = \mathcal{L}_\xi T^{\lambda}{}_{\mu\nu} = \mathcal{L}_\xi Q^{\lambda}{}_{\mu\nu} = 0 \implies \mathcal{L}_\xi \tilde{R}^{\lambda}{}_{\rho\mu\nu} = 0\,. \tag{9}$$

- Stationary and axisymmetric space-times[4]:

$$\#10 \to \#4 \begin{cases} ds^2 = \Psi_1(r,\vartheta)\,dt^2 - \frac{dr^2}{\Psi_2(r,\vartheta)} \\ -\,r^2\Psi_3(r,\vartheta)\Big[d\vartheta^2 + \sin^2\vartheta(d\varphi - \Psi_4(r,\vartheta)dt)^2\Big] \end{cases} ;$$

$$\#24\left\{T^{\lambda}{}_{\mu\nu} = T^{\lambda}{}_{\mu\nu}(r,\vartheta)\right. \tag{10}$$

$$\#4\left\{W_\mu = (W_t(r,\vartheta), W_r(r,\vartheta), W_\vartheta(r,\vartheta), W_\varphi(r,\vartheta))\,.\right. \tag{11}$$

[4] S. Bahamonde and J. G. Valcarcel, arXiv: 2108.12414, to be published in JCAP.

Extension to axisymmetric space-times

- Rotating Kerr-Newman metric structure:

$$ds^2 = \Psi(r,\vartheta)\,dt^2 - \frac{r^2 + a^2\cos^2\vartheta}{(r^2 + a^2\cos^2\vartheta)\,\Psi(r,\vartheta) + a^2\sin^2\vartheta}\,dr^2 - \left(r^2 + a^2\cos^2\vartheta\right)d\vartheta^2 + 2a\left(1 - \Psi(r,\vartheta)\right)\sin^2\vartheta\,dtd\varphi - \sin^2\vartheta\left[r^2 + a^2 + a^2\left(1 - \Psi(r,\vartheta)\right)\sin^2\vartheta\right]d\varphi^2\,, \tag{12}$$

$$\Psi(r,\vartheta) = 1 - \frac{\left[2mr + 4e_1(\kappa^2_{\mathrm{d,e}} + \kappa^2_{\mathrm{d,m}}) - d_1\kappa^2_{\mathrm{s}}\right]}{r^2 + a^2\cos^2\vartheta}\,. \tag{13}$$

Extension to axisymmetric space-times

- Rotating Kerr-Newman metric structure:

$$\begin{aligned} ds^2 &= \Psi(r,\vartheta)\, dt^2 - \frac{r^2 + a^2\cos^2\vartheta}{(r^2+a^2\cos^2\vartheta)\,\Psi(r,\vartheta) + a^2\sin^2\vartheta}\, dr^2 \\ &- \left(r^2 + a^2\cos^2\vartheta\right) d\vartheta^2 + 2a\left(1-\Psi(r,\vartheta)\right)\sin^2\vartheta\, dt d\varphi \\ &- \sin^2\vartheta\left[r^2 + a^2 + a^2\left(1-\Psi(r,\vartheta)\right)\sin^2\vartheta\right] d\varphi^2\,, \end{aligned} \quad (12)$$

$$\Psi(r,\vartheta) = 1 - \frac{\left[2mr + 4e_1(\kappa^2_{\mathrm{d,e}} + \kappa^2_{\mathrm{d,m}}) - d_1\kappa^2_{\mathrm{s}}\right]}{r^2 + a^2\cos^2\vartheta}\,. \quad (13)$$

- Field strength tensors:

$$\begin{aligned} \bar{R}_{[\mu\nu]} &= \frac{1}{12}\varepsilon^{\lambda}{}_{\sigma\mu\nu}\nabla_\lambda \bar{S}^\sigma + \frac{1}{2}\nabla_\lambda \bar{t}^{\lambda}{}_{\mu\nu}\,; \quad \tilde{R}^{\lambda}{}_{\lambda\mu\nu} = 4\nabla_{[\nu}W_{\mu]}\,; \\ \bar{R}^{\lambda}{}_{[\mu\nu\rho]} &= \frac{1}{6}\varepsilon^{\lambda}{}_{\sigma[\rho\nu}\nabla_{\mu]}\bar{S}^\sigma + \nabla_{[\mu}\bar{t}^{\lambda}{}_{\rho\nu]} + \frac{1}{4}\varepsilon^{\lambda}{}_{\omega\sigma[\rho}\mathring{t}_1^{\sigma}{}_{\mu\nu]}\bar{S}^\omega \\ &- \frac{1}{18}\varepsilon_{\sigma\mu\nu\rho}\mathring{T}_1^{\lambda}\bar{S}^\sigma\,. \end{aligned} \quad (14)$$

Extension to axisymmetric space-times

- Nonmetricity sector:

$$w_1(r,\vartheta) = \frac{\kappa_{\rm d,e} r - a\,\kappa_{\rm d,m}\cos\vartheta}{r^2 + a^2\cos^2\vartheta}\,, \quad w_3(r,\vartheta) = 0\,,$$
$$w_2(r,\vartheta) = -\frac{\kappa_{\rm d,e} r}{(r^2 + a^2\cos^2\vartheta)\Psi(r,\vartheta) + a^2\sin^2\vartheta}\,,$$
$$w_4(r,\vartheta) = \kappa_{\rm d,m}\left(\frac{r^2+a^2}{r^2 + a^2\cos^2\vartheta}\cos\vartheta - \gamma\right) - a\,\frac{\kappa_{\rm d,e} r\sin^2\vartheta}{r^2 + a^2\cos^2\vartheta}\,. \qquad (15)$$

Extension to axisymmetric space-times

- Nonmetricity sector:

$$w_1(r,\vartheta) = \frac{\kappa_{\rm d,e} r - a\,\kappa_{\rm d,m}\cos\vartheta}{r^2 + a^2\cos^2\vartheta}\,, \quad w_3(r,\vartheta) = 0\,,$$
$$w_2(r,\vartheta) = -\frac{\kappa_{\rm d,e} r}{(r^2 + a^2\cos^2\vartheta)\Psi(r,\vartheta) + a^2\sin^2\vartheta}\,,$$
$$w_4(r,\vartheta) = \kappa_{\rm d,m}\left(\frac{r^2+a^2}{r^2+a^2\cos^2\vartheta}\cos\vartheta - \gamma\right) - a\,\frac{\kappa_{\rm d,e} r\sin^2\vartheta}{r^2+a^2\cos^2\vartheta}\,. \qquad (15)$$

- Torsion sector (decoupling limit between the spin and the orbital angular momentum $|a\kappa_{\rm s}| \ll 1$):

$$\bar{\mathcal{S}}^a = -\frac{\kappa_{\rm s}}{r}(1,1,0,0) + \mathcal{O}(a\kappa_{\rm s})\,, \qquad (16)$$

$$\bar{\mathcal{T}}_2^{abc} = \frac{\kappa_{\rm s}}{3r}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} + \mathcal{O}(a\kappa_{\rm s})\,. \qquad (17)$$

Gravitational spin-orbit interaction

- We found a solution in the decoupling limit $a\kappa_s \ll 1$, which ensures that the Maxwell equation and closure conditions are fulfilled by the field strength tensors of torsion

$$\nabla_\lambda \tilde{R}^\lambda{}_{[\rho\mu\nu]} \quad = \quad \nabla_\mu \tilde{R}^{[\mu\nu]} = 0\,, \quad \nabla_{[\sigma} \tilde{R}_{\lambda[\rho\mu\nu]]} = \nabla_{[\lambda} \tilde{R}_{[\mu\nu]]} = 0\,.$$

Gravitational spin-orbit interaction

- We found a solution in the decoupling limit $a\kappa_s \ll 1$, which ensures that the Maxwell equation and closure conditions are fulfilled by the field strength tensors of torsion

$$\nabla_\lambda \tilde{R}^\lambda{}_{[\rho\mu\nu]} = \nabla_\mu \tilde{R}^{[\mu\nu]} = 0\,, \quad \nabla_{[\sigma}\tilde{R}_{\lambda[\rho\mu\nu]]} = \nabla_{[\lambda}\tilde{R}_{[\mu\nu]]} = 0\,.$$

Gravitational spin-orbit interaction

- We found a solution in the decoupling limit $a\kappa_s \ll 1$, which ensures that the Maxwell equation and closure conditions are fulfilled by the field strength tensors of torsion

$$\nabla_\lambda \tilde{R}^\lambda{}_{[\rho\mu\nu]} = \nabla_\mu \tilde{R}^{[\mu\nu]} = 0\,, \quad \nabla_{[\sigma}\tilde{R}_{\lambda[\rho\mu\nu]]} = \nabla_{[\lambda}\tilde{R}_{[\mu\nu]]} = 0\,.$$

Possible new effects in the decoupling limit

The dynamics of torsion and nonmetricity alters the geometry of the space-time $\Longrightarrow$

Gravitational spin-orbit interaction

- We found a solution in the decoupling limit $a\kappa_s \ll 1$, which ensures that the Maxwell equation and closure conditions are fulfilled by the field strength tensors of torsion

$$\nabla_\lambda \tilde{R}^\lambda{}_{[\rho\mu\nu]} = \nabla_\mu \tilde{R}^{[\mu\nu]} = 0\,, \quad \nabla_{[\sigma}\tilde{R}_{\lambda[\rho\mu\nu]]} = \nabla_{[\lambda}\tilde{R}_{[\mu\nu]]} = 0\,.$$

Possible new effects in the decoupling limit

The dynamics of torsion and nonmetricity alters the geometry of the space-time $\Longrightarrow$

Gravitational spin-orbit interaction

- We found a solution in the decoupling limit $a\kappa_s \ll 1$, which ensures that the Maxwell equation and closure conditions are fulfilled by the field strength tensors of torsion

$$\nabla_\lambda \tilde{R}^\lambda{}_{[\rho\mu\nu]} = \nabla_\mu \tilde{R}^{[\mu\nu]} = 0\,, \quad \nabla_{[\sigma}\tilde{R}_{\lambda[\rho\mu\nu]]} = \nabla_{[\lambda}\tilde{R}_{[\mu\nu]]} = 0\,.$$

Possible new effects in the decoupling limit

The dynamics of torsion and nonmetricity alters the geometry of the space-time $\Longrightarrow$ Additional modifications provided by a strong coupling between the orbital and the spin angular.

Gravitational spin-orbit interaction

- We found a solution in the decoupling limit $a\kappa_s \ll 1$, which ensures that the Maxwell equation and closure conditions are fulfilled by the field strength tensors of torsion

$$\nabla_\lambda \tilde{R}^\lambda{}_{[\rho\mu\nu]} \;=\; \nabla_\mu \tilde{R}^{[\mu\nu]} = 0\,, \quad \nabla_{[\sigma}\tilde{R}_{\lambda[\rho\mu\nu]]} = \nabla_{[\lambda}\tilde{R}_{[\mu\nu]]} = 0\,.$$

Possible new effects in the decoupling limit

The dynamics of torsion and nonmetricity alters the geometry of the space-time $\Longrightarrow$ Additional modifications provided by a strong coupling between the orbital and the spin angular.

Gravitational spin-orbit interaction

- We found a solution in the decoupling limit $a\kappa_s \ll 1$, which ensures that the Maxwell equation and closure conditions are fulfilled by the field strength tensors of torsion

$$\nabla_\lambda \tilde{R}^\lambda{}_{[\rho\mu\nu]} = \nabla_\mu \tilde{R}^{[\mu\nu]} = 0\,, \quad \nabla_{[\sigma}\tilde{R}_{\lambda[\rho\mu\nu]]} = \nabla_{[\lambda}\tilde{R}_{[\mu\nu]]} = 0\,.$$

Possible new effects in the decoupling limit

The dynamics of torsion and nonmetricity alters the geometry of the space-time $\Longrightarrow$ Additional modifications provided by a strong coupling between the orbital and the spin angular.

- **Gravitational spin-orbit interaction:**

$$\mathcal{H}_{\rm LS} = \frac{1}{m_{\rm e}^2 r}\frac{\partial V}{\partial r}\mathbf{L}\cdot\mathbf{S} \approx \frac{d_1}{2r}\frac{\partial g_{tt}}{\partial r}a\kappa_{\rm s}\cos\vartheta \qquad (18)$$

Conclusions

- In the 1st paper we found an exact black hole solution in a MAG theory with torsion and nonmetricity being dynamical and independent.

Conclusions

- In the 1st paper we found an exact black hole solution in a MAG theory with torsion and nonmetricity being dynamical and independent.
- In the 2nd paper, we studied the phenomenology of particle and provided observational constrains for the charges.

Conclusions

- In the 1st paper we found an exact black hole solution in a MAG theory with torsion and nonmetricity being dynamical and independent.
- In the 2nd paper, we studied the phenomenology of particle and provided observational constrains for the charges.
- In the 3rd paper, we obtained an axially symmetric solution behaving as a Kerr-Newman-de Sitter solution (in the decoupling limit).

Conclusions

- In the 1st paper we found an exact black hole solution in a MAG theory with torsion and nonmetricity being dynamical and independent.
- In the 2nd paper, we studied the phenomenology of particle and provided observational constrains for the charges.
- In the 3rd paper, we obtained an axially symmetric solution behaving as a Kerr-Newman-de Sitter solution (in the decoupling limit).
- In progress: we are constructing Plebanski-Demianski uniformly accelerated rotating black hole solutions with NUT parameter, electromagnetic charges and a cosmological constant.

Conclusions

- In the 1st paper we found an exact black hole solution in a MAG theory with torsion and nonmetricity being dynamical and independent.
- In the 2nd paper, we studied the phenomenology of particle and provided observational constrains for the charges.
- In the 3rd paper, we obtained an axially symmetric solution behaving as a Kerr-Newman-de Sitter solution (in the decoupling limit).
- In progress: we are constructing Plebanski-Demianski uniformly accelerated rotating black hole solutions with NUT parameter, electromagnetic charges and a cosmological constant.
- Future: search of a gravitational spin-orbit interaction in MAG beyond the Kerr-Newman space-time (MAG is the main candidate to describe a spin-orbit interaction beyond GR).