# Distinctive Features of Hairy Black Holes in Teleparallel gravity

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In collaboration with Christian Pfeifer, Daniel Doneva, Stoytcho S. Yazadjiev and Ludovic Ducobu.

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# Overview of the Talk

# Curvature tensor

- The curvature becomes

$$\tilde{R}^{\mu}{}_{\nu\rho\sigma} = \mathring{R}^{\mu}{}_{\nu\rho\sigma} + \mathring{\nabla}_{\rho}\tilde{D}^{\mu}{}_{\nu\sigma} - \mathring{\nabla}_{\sigma}\tilde{D}^{\mu}{}_{\nu\rho} + \tilde{D}^{\mu}{}_{\tau\rho}\tilde{D}^{\tau}{}_{\nu\sigma} - \tilde{D}^{\mu}{}_{\tau\sigma}\tilde{D}^{\tau}{}_{\nu\rho}\,.$$

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with

$$T := T^{\rho\lambda\kappa}T_{\rho\lambda\kappa} + 2T^{\rho\lambda\kappa}T_{\kappa\rho\lambda} - 4T_{\rho}{}^{\kappa}{}_{\kappa}T^{\rho\lambda}{}_{\lambda}\,, \quad \text{Torsion scalar}\,,$$

$$Q := -\frac{1}{4}Q_{\alpha\beta\gamma}Q^{\alpha\beta\gamma} + \frac{1}{2}Q_{\alpha\beta\gamma}Q^{\beta\alpha\gamma} + \frac{1}{4}Q_{\alpha}Q^{\alpha} - \frac{1}{2}Q_{\alpha}\bar{Q}^{\alpha}\,, \ \text{Nonmetricity scalar}\,,$$

$$C := 2(Q_{\kappa\rho\lambda}T^{\lambda\kappa\rho} + Q_{\rho}{}^{\sigma}{}_{\sigma}T^{\rho\kappa}{}_{\kappa} - Q^{\sigma}{}_{\sigma\rho}T^{\rho\kappa}{}_{\kappa})\,.$$

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**(torsional) Teleparallel equivalent of GR (TEGR) action**

$$S_{\rm TEGR} = \int \left[-\frac{1}{2\kappa^2}T + L_{\rm m}\right] e\, d^4x .$$

- Since $\mathring{R}$ differs by $T$ by a boundary term $B$, **the equations of TEGR are equivalent to the Einstein's field equations.**

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2. Generic properties of Teleparallel Theories
3. Teleparallel scalar Gauss-Bonnet gravity
   - Scalar-Gauss Bonnet gravity
   - Teleparallel Gauss-Bonnet

# Connection in Teleparallel gravity

- GR assumes zero torsion and non-zero curvature (Levi-Civita connection) whereas Teleparallel gravity (TG) assumes **zero curvature and non-zero torsion**. Both assumes non-metricity to be zero, meaning $\tilde{\nabla}_\alpha g_{\mu\nu} = 0$.

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### Torsion tensor

$$T^\rho{}_{\mu\nu} = \Gamma^\rho{}_{\nu\mu} - \Gamma^\rho{}_{\mu\nu} = E_A{}^\rho \left( e^A{}_{\nu,\mu} - e^A{}_{\mu,\nu} + \omega^A{}_{B\mu} e^B{}_\nu - \omega^A{}_{B\nu} e^B{}_\mu \right).$$

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# Important properties of Teleparallel theories

- Variations of any action should be taken with respect to both variables (with the emphasis that the spin connection is non-arbitrary but always flat).

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- Since $\omega^{A}{}_{B\mu}$ is a pure-gauge quantity, it can be shown that the the antisymmetric part of the field equations arising from variations w/r to the tetrads $e^{A}{}_{\mu}$ coincides with the variations of the action w/r to $\omega^{A}{}_{B\mu}$.

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- Then, in the Weitzenbock gauge (zero spin connection), it is sufficient to: [2]

$$\delta_e S \Longrightarrow E_{(\mu\nu)} = \kappa^2 \Theta_{\mu\nu}\,, \quad E_{[\mu\nu]} = 0\,, \tag{1}$$

then, we have $10+6$ dof of the tetrad in the symmetric+antisymmetric field equations.

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# Spherical Symmetry in Teleparallel gravity

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# Spherical Symmetry in Teleparallel gravity

- In TG, all the dynamics can be put in the tetrad and the spin connection can be set to be zero.
- We assume that the connection and metric have the same symmetries:

$$\mathcal{L}_{Z_\zeta} e^A{}_\mu = -\lambda_\zeta^A{}_B e^B{}_\mu \,, \quad \mathcal{L}_{Z_\zeta} \omega^A{}_{B\mu} = \partial_\mu \lambda_\zeta^A{}_B + \omega^A{}_{C\mu} \lambda_\zeta^C{}_B - \omega^C{}_{B\mu} \lambda_\zeta^A{}_C \,. \tag{2}$$

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- The most general tetrad satisfying spherical symmetry in the Weitzenbock gauge (zero spin connection) is [3]

$$e^A{}_\nu = \begin{pmatrix} C_1 & C_2 & 0 & 0 \\ C_3 \sin\theta\cos\phi & C_4\sin\theta\cos\phi & C_5\cos\theta\cos\phi - C_6\sin\phi & -\sin\theta(C_5\sin\phi + C_6\cos\theta\cos\phi) \\ C_3\sin\theta\sin\phi & C_4\sin\theta\sin\phi & C_5\cos\theta\sin\phi + C_6\cos\phi & \sin\theta(C_5\cos\phi - C_6\cos\theta\sin\phi) \\ C_3\cos\theta & C_4\cos\theta & -C_5\sin\theta & C_6\sin^2\theta \end{pmatrix},$$

where $C_i = C_i(t,r)$.

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where $C_i = C_i(t,r)$.

- Using $g_{\mu\nu} = \eta_{AB} e^A{}_\mu e^B{}_\nu$, we have that the metric is

$$\begin{aligned} \mathrm{d}s^2 \quad = \quad & \left(C_1^2 - C_3^2\right)\mathrm{d}t^2 - 2(C_3C_4 - C_1C_2)\,\mathrm{d}t\,\mathrm{d}r - \left(C_4^2 - C_2^2\right)\mathrm{d}r^2 \\ & - \left(C_5^2 + C_6^2\right)\left(\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2\right), \end{aligned}$$

where we have cross-terms.

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# Spherical Symmetry in Teleparallel gravity

- Without losing generality, we can choose a coordinate system such that the cross term vanishes. This can be easily done by taking the following reparametrization:

$$\begin{aligned} C_1(r) &= \nu A(r)\cosh\beta(r)\,, \quad C_3(r) = \nu A(r)\sinh\beta(r)\,, \\ C_4(r) &= \xi B(r)\cosh\beta(r)\,, \quad C_2(r) = \xi B(r)\sinh\beta(r)\,, \\ C_5(r) &= \chi C(r)\cos\alpha(r)\,, \quad C_6(r) = \chi C(r)\sin\alpha(r)\,, \end{aligned}$$

with $\{\nu,\xi,\chi\}$ being $\pm 1$. This tetrad gives the metric in the standard form in spherical coordinates:

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- Note that $\beta(r), \alpha(r)$ are tetrad dof (they do not appear in the metric). They can be set by solving the antisymmetric field equations.

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- This is a theorem, meaning that in GR, this is the unique asymptotically flat black hole solution. In spherical symmetry, it is just Schwarzschild.
- Is it possible that black holes have hair? One needs to go beyond GR, either by having modified gravity or allowing extra degrees of freedom such as scalar fields coupled to gravity.

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$$\mathring{G} = \mathring{R}_{\alpha\beta\mu\nu}\mathring{R}^{\alpha\beta\mu\nu} - 4\mathring{R}_{\alpha\beta}\mathring{R}^{\alpha\beta} + \mathring{R}^2\,.$$

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- However, if one allows couplings between the Gauss-Bonnet invariant and a scalar field, then the field equations will not be longer equivalent to GR.

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- The Scalar Gauss-Bonnet (sGB) gravity theory is described by the following action

$$\mathcal{S}_{\rm sGB} = \frac{1}{2\kappa^2}\int \left[\mathring{R} - \frac{1}{2}\beta\,\partial_\mu\psi\partial^\mu\psi + \alpha\mathcal{G}(\psi)\mathring{G}\right]\sqrt{-g}\,{\rm d}^4x\,.$$

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- If $\mathcal{G}(\psi) = 1$, then we just have GR. Equivalently, if $\psi = \mathrm{const}$, we just have GR.

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$$\mathcal{S}_{\rm sGB} = \frac{1}{2\kappa^2}\int \left[\mathring{R} - \frac{1}{2}\beta\,\partial_\mu\psi\partial^\mu\psi + \alpha\mathcal{G}(\psi)\mathring{G}\right]\sqrt{-g}\,{\rm d}^4x\,.$$

- If $\mathcal{G}(\psi) = 1$, then we just have GR. Equivalently, if $\psi = \mathrm{const}$, we just have GR.
- For $\mathcal{G}(\psi) \neq \mathrm{const}$, the field equations are not longer GR. For example the scalar-field equation is

$$\beta\mathring{\Box}\psi + \alpha\dot{\mathcal{G}}(\psi)\mathring{G} = 0\,, \tag{3}$$

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  where $\dot{\mathcal{G}}(\psi) = d\mathcal{G}/d\psi$.
- Note: $\overset{\circ}{R} = 0$ in Schwarzschild but $\overset{\circ}{G} \neq 0$. That property would be important to understand the difference between this model and the former one.

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- It has been shown (numerically) that for some particular coupling functions, there are asymptotically flat scalarized black hole solutions where the scalar charge emerges from a mechanism called **Spontaneous scalarization**.

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- Thus the solution bifurcates to a different black hole solution with scalar hair.
- This transition is usually smooth in sGB and shares similarities with second order phase transitions.

# Scalar fields non-minimally coupled to Torsion

- In our previous paper[4], we studied Teleparallel theories with a scalar field, for example[5]:

$$S = \frac{1}{2\kappa^2}\int_M \left[-\mathcal{A}(\psi)T - \tilde{\mathcal{C}}(\psi)B - \frac{1}{2}\mathcal{B}(\psi)\partial_\mu\psi\partial^\mu\psi - 2\kappa^2\mathcal{V}(\psi)\right]\sqrt{-g}\,\mathrm{d}^4x\,,$$

where $X = -\frac{1}{2}g^{\mu\nu}\partial_\mu\psi\partial_\nu\psi$.

[4] S. Bahamonde, L. Ducobu and C. Pfeifer, JCAP **04** (2022) no.04, 018

[5] S. Bahamonde and M. Wright, Phys. Rev. D **92** (2015) no.8, 084034; M. Zubair, S. Bahamonde and M. Jamil, Eur. Phys. J. C **77** (2017) no.7, 472; M. Hohmann and C. Pfeifer, Phys. Rev. D **98** (2018) no.6, 064003.

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- We found new exact black hole solutions (some of them different to the Riemannian case), but they seem to be not so much interesting phenomenologically.

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Here, $D_\lambda$ is the cov derivative of the Tele connection and $K^{\rho}{}_{\mu\nu} = \Gamma^{\rho}{}_{\mu\nu} - \mathring{\Gamma}^{\rho}{}_{\mu\nu} = \frac{1}{2}\left(T_{\mu}{}^{\rho}{}_{\nu} + T_{\nu}{}^{\rho}{}_{\mu} - T^{\rho}{}_{\mu\nu}\right)$.

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- Then, two Teleparallel Gauss-Bonnet invariants appear in the Teleparallel framework. $T_G$ is a topological invariant in $4D$ and $B_G$ is a boundary term (in all dimensions).

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$$\mathcal{S}_{\rm TsGB} = \frac{1}{2\kappa^2}\int \Big[ -T - \frac{1}{2}\beta\,\partial_\mu\psi\partial^\mu\psi + \alpha_1\mathcal{G}_1(\psi)T_G + \alpha_2\mathcal{G}_2(\psi)B_G \Big]\, e\,\mathrm{d}^4x\,,$$

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- The second and third cases ($\alpha_3 \neq 0$) are new in the literature and they can only exist when one considers Teleparallel gravity.

# Antisymmetric field equations - Solutions

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- Tetrad contains more dof than the metric (two different tetrads can give rise the same metric - Lorentz dof).
- The first branch which solves the antisymmetric equations is $\beta(r) = i\pi\, n_1\,, \alpha(r) = \pi\, n_2$ which gives

$$e^{(1)\,a}{}_{\mu} = \begin{pmatrix} \nu A & 0 & 0 & 0 \\ 0 & \xi B \sin\theta\cos\phi & \chi C \cos\theta\cos\phi & -\chi C \sin\theta\sin\phi \\ 0 & \xi B \sin\theta\sin\phi & \chi C \cos\theta\sin\phi & \chi C \sin\theta\cos\phi \\ 0 & \xi B\cos\theta & -\chi C\sin\theta & 0 \end{pmatrix}\,, \quad \{\nu,\xi,\chi\} = \pm 1\,.$$

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- The second branch branch which solves the antisymmetric equations is $\beta(r) = \frac{i\pi}{2} + i\pi\, n_3\,, \alpha(r) = \frac{\pi}{2} + \pi$ which gives

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- The theory gives us the possibility to have such an unstable mode if

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- For our theory, spontaneous scalarization can occur in our theory for a much larger choice of parameters for different masses.
- Contrary to the sGB case, where scalarization of non-rotating black holes was possible only for $\alpha_2<0$, we can have scalarization for different signs of the coupling parameters.

- When $\alpha_3\dot{\mathcal{G}}_3 \neq \alpha_2\dot{\mathcal{G}}_2$, the scalar field at the horizon must satisfy [7]

$$\psi'_H = \frac{r_H}{4(\alpha_2\dot{\mathcal{G}}_2-\alpha_3\dot{\mathcal{G}}_3)}\left(1\pm\frac{1}{\beta}\left[\beta^2+\frac{32(\alpha_3\dot{\mathcal{G}}_3-\alpha_2\dot{\mathcal{G}}_2)}{r_H^8}\left\{32\alpha_3^2\dot{\mathcal{G}}_3^2(\alpha_3\dot{\mathcal{G}}_3-\alpha_2\dot{\mathcal{G}}_2)\right.\right.\right.$$
$$\left.\left.\left.+\beta r_H^4(3\alpha_2\dot{\mathcal{G}}_2+\alpha_3\dot{\mathcal{G}}_3)\right\}\right]^{1/2}\right)-\frac{8\alpha_3\dot{\mathcal{G}}_3}{\beta r_H^3}\,.$$

[7] S. Bahamonde, D. D. Doneva, L. Ducobu, C. Pfeifer and S. S. Yazadjiev, Phys. Rev. D **107** (2023) no.10, 104013

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$$\begin{aligned}\psi'_H \quad &= \quad \frac{r_H}{4(\alpha_2\dot{\mathcal{G}}_2 - \alpha_3\dot{\mathcal{G}}_3)}\Bigg(1 \pm \frac{1}{\beta}\Big[\beta^2 + \frac{32(\alpha_3\dot{\mathcal{G}}_3 - \alpha_2\dot{\mathcal{G}}_2)}{r_H^8}\Big\{32\alpha_3^2\dot{\mathcal{G}}_3^2(\alpha_3\dot{\mathcal{G}}_3 - \alpha_2\dot{\mathcal{G}}_2) \\ &\quad +\beta r_H^4(3\alpha_2\dot{\mathcal{G}}_2 + \alpha_3\dot{\mathcal{G}}_3)\Big\}\Big]^{1/2}\Bigg) - \frac{8\alpha_3\dot{\mathcal{G}}_3}{\beta r_H^3}\,.\end{aligned}$$

- This branch gives the correct condition for the Riemannian sGB case ($\alpha_3 = 0$) that has been used widely in the literature to solving the equations numerically:

$$\psi'_H \quad = \quad \frac{1}{4\alpha_2 r_H\dot{\mathcal{G}}_2}\left[r_H^2 \pm \sqrt{r_H^4 - \frac{96\alpha_2^2\dot{\mathcal{G}}_2^2}{\beta}}\right].$$

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- This analysis suggests that there are two different branches in Teleparallel sGB having asymptotically flat scalarized black hole configurations.

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- By Taking $\beta = 4$ (kinetic constant), and setting the background value of the scalar field to zero, we have two coupling constants $\alpha_2$ and $\alpha_3$ and two coupling functions $\mathcal{G}_2$ and $\mathcal{G}_3$.

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- We have decided to fix the two coupling functions in the following form

$$\mathcal{G}_2(\psi) = \frac{1}{12}\left(1 - e^{-6\psi^2}\right) = \mathcal{G}_3(\psi)\,.$$

This exponential function has one of the desired properties for scalarization, namely, it allows the GR solutions with a zero scalar field to be also solutions of the more general system of equations in Teleparallel gravity.

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  This exponential function has one of the desired properties for scalarization, namely, it allows the GR solutions with a zero scalar field to be also solutions of the more general system of equations in Teleparallel gravity.
- For this coupling, it was proven [8] that stable scalarized black hole solutions exist in the sGB case.

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- Having fixed $\mathcal{G}_2$ and $\mathcal{G}_3$, the only theory parameters left to vary are $\alpha_2$ and $\alpha_3$ and more precisely, their relative weight. The following two cases are especially interesting, since they are purely Teleparallel, i.e. *the scalarization is triggered by torsion*:

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  In either of these two limiting cases, no contribution of the Riemannian Gauss-Bonnet term is present.
- These cases go beyond the classification of theories allowing for scalarization that is discussed in a recent Review[9]

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# Numerical solutions - Mass and scalar charge case 1

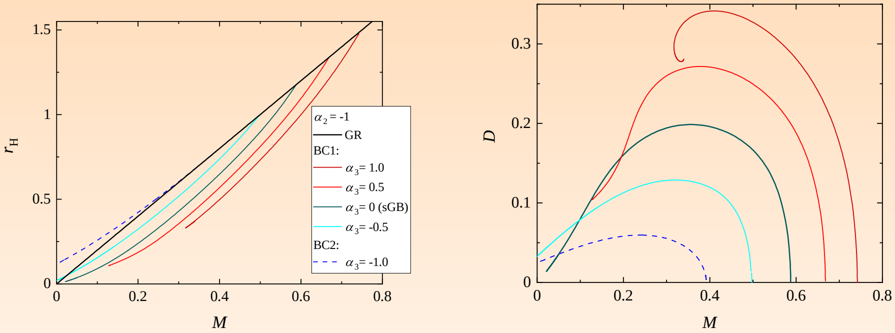


Figure: Setting $\alpha_2 = -1$ (the Riemannian sGB) and varying $\alpha_3$

- With the increase of $\alpha_3$ the point of bifurcation from the GR branch moves to large masses.

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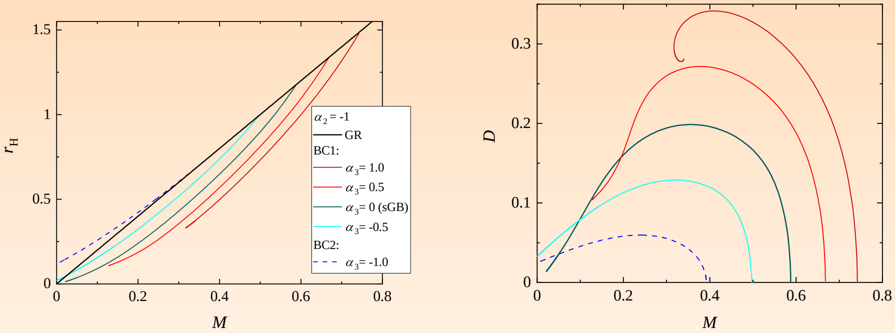


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- For larger $\alpha_3$, the branch of scalarized solutions disappears at smaller masses.

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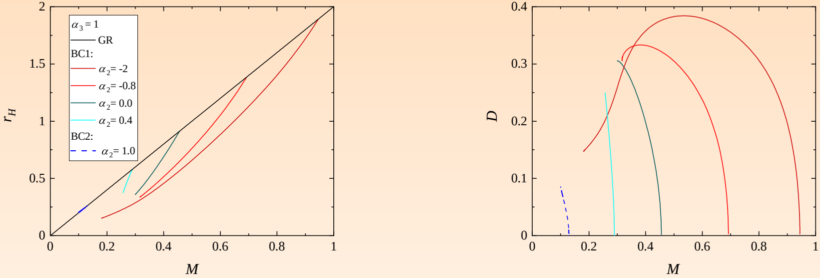


Figure: Setting $\alpha_3 = 1$ (the Teleparallel part) and varying $\alpha_2$

- Contrary to the previous figure, larger $\alpha_2$ move the bifurcation point to smaller masses

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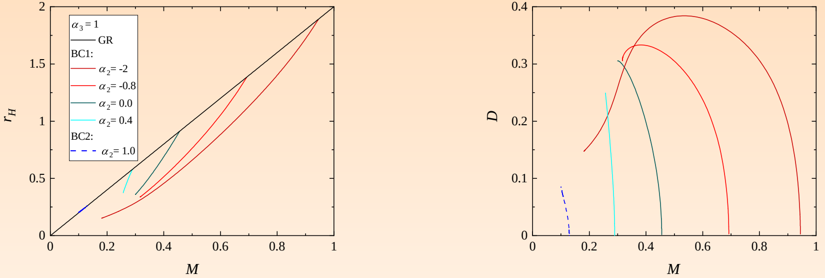


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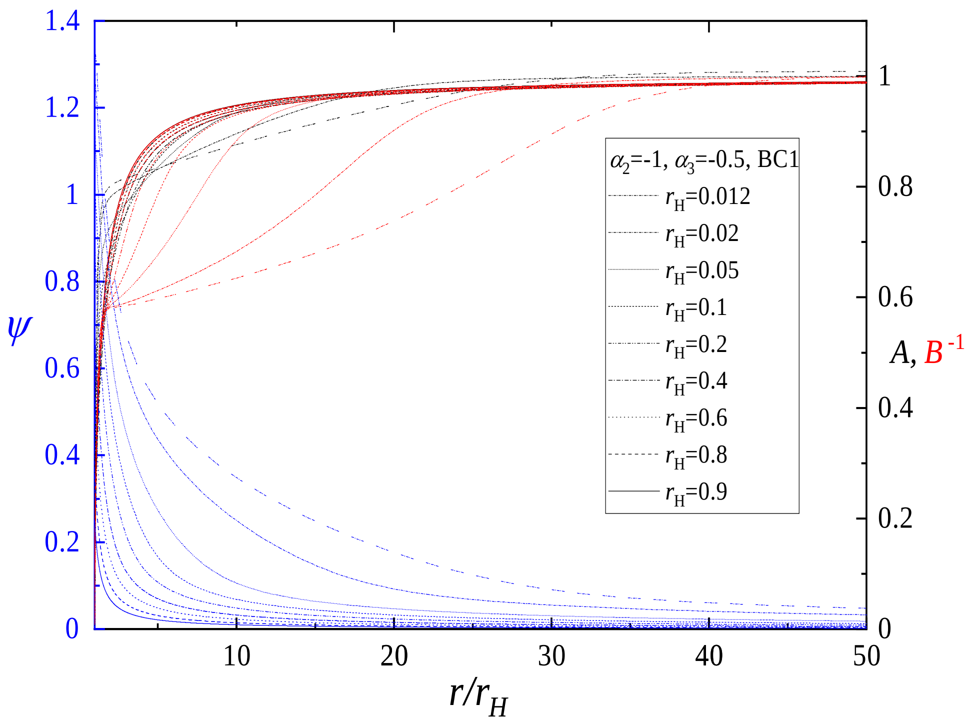
- Even though this case offers a completely new type of scalarization, the behaviour of the solutions branches is qualitatively very similar to the sGB theory



1.4
1.2
1
0.8
0.6
0.4
0.2
0
ψ
10
20
30
40
50
r/r_H
1
0.8
0.6
0.4
0.2
0
A, B^-1
α_2=-1, α_3=-0.5, BC1
r_H=0.012
r_H=0.02
r_H=0.05
r_H=0.1
r_H=0.2
r_H=0.4
r_H=0.6
r_H=0.8
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# New results with different couplings

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  - $\mathcal{G}_i = \psi^2$, which leads to black hole scalarization, i.e. Schwarzschild black hole is always a solution of the field equations but for small black hole masses, it becomes unstable giving rise to a spontaneously scalarized branch of solutions.
- Even though simpler compared to the exponential coupling considered before, the second choice leads to unstable black hole solutions in the Riemannian Gauss-Bonnet case. Interestingly, this observation might change for a strong enough torsional contribution.

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  4. This has interesting implications: For example, if put in a binary, such a black hole will emit only very little scalar dipole radiation while the scalar field might influence the binary dynamics significantly.

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  3. our results indicate that the pure teleparallel term might potentially lead to a stabilization of the black holes for pure quadratic coupling.

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  4. Within Symmetric TG, can one construct similar scalarization with the new Gauss-Bonnet that we derived?