Scalarized Black Holes in Teleparallel Gravity

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Outline

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Fundamental variables and characteristic tensors

- In the most general metric-affine setting, the fundamental variables are a **metric** $g_{\mu\nu}$ (10 comp.) as well as the coefficients $\hat{\Gamma}^{\rho}{}_{\mu\nu}$ (64 comp.) of an **affine connection**.

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Curvature	$\hat{R}^{\mu}{}_{\nu\rho\sigma} = \partial_{\rho}\hat{\Gamma}^{\mu}{}_{\nu\sigma} - \partial_{\sigma}\hat{\Gamma}^{\mu}{}_{\nu\rho} + \hat{\Gamma}^{\mu}{}_{\tau\rho}\hat{\Gamma}^{\tau}{}_{\nu\sigma} - \hat{\Gamma}^{\mu}{}_{\tau\sigma}\hat{\Gamma}^{\tau}{}_{\nu\rho}$
Torsion	$\hat{T}^{\mu}{}_{\nu\rho} = \hat{\Gamma}^{\mu}{}_{\nu\rho} - \hat{\Gamma}^{\mu}{}_{\rho\nu}$
Nonmetricity	$\hat{Q}_{\mu\nu\rho} = \hat{\nabla}_{\mu}g_{\nu\rho} = \partial_{\mu}g_{\nu\rho} - \hat{\Gamma}^{\sigma}{}_{\nu\mu}g_{\sigma\rho} - \hat{\Gamma}^{\sigma}{}_{\rho\mu}g_{\nu\sigma}$

Tetrads and spin connection

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- Tetrads satisfy the **orthogonality condition**; $E_m{}^\mu e^n{}_\mu = \delta^n_m$ and $E_m{}^\nu e^m{}_\mu = \delta^\nu_\mu$ and the metric and its inverse can be reconstructed via

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where η_{ab} is the Minkowski metric.

Teleparallel connection choice

Weitzenböck connection: Curvature is zero and $\nabla_\mu g_{\nu\rho} = 0$:

Teleparallel connection - Weitzenböck

$$\tilde{\Gamma}^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\mu\nu} + \frac{1}{2} T^{\lambda}{}_{\mu\nu} - T_{(\mu}{}^{\lambda}{}_{\nu)} + \cancel{\frac{1}{2} Q^{\lambda}{}_{\mu\nu} - Q_{(\mu}{}^{\lambda}{}_{\nu)}} \equiv 0\,.$$

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$$\Gamma^{\rho}{}_{\mu\nu} = E_a{}^{\rho} D_\mu e^a{}_\nu = E_a{}^{\rho}(\partial_\mu e^a{}_\nu + w^a{}_{b\mu} e^b{}_\nu)\,,$$

$$\Longrightarrow R^a{}_{b\mu\nu}(\omega^a{}_{b\mu}) = \partial_\mu \omega^a{}_{b\nu} - \partial_\nu \omega^a{}_{b\mu} + \omega^a{}_{c\mu}\omega^c{}_{b\nu} - \omega^a{}_{c\nu}\omega^c{}_{b\mu} \equiv 0\,.$$

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In TG, it is always possible to find a frame such that $\omega^a{}_{b\mu} = 0$, but this is a gauge choice, so only some tetrads are compatible with this.

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$$T^{\rho}{}_{\mu\nu} = \Gamma^{\rho}{}_{\nu\mu} - \Gamma^{\rho}{}_{\mu\nu} = E_A{}^{\rho}\left(e^{A}{}_{\nu,\mu} - e^{A}{}_{\mu,\nu} + \omega^{A}{}_{B\mu}e^{B}{}_{\nu} - \omega^{A}{}_{B\nu}e^{B}{}_{\mu}\right).$$

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- The torsion tensor is generally non-vanishing, and transforms covariantly under both diffeomorphisms and local Lorentz transformations.

Ricci scalar and torsion scalar

- One has that
$\Gamma^{\rho}{}_{\nu\mu} = \mathring{\Gamma}^{\rho}{}_{\nu\mu} + \frac{1}{2}(T_{\mu}{}^{\rho}{}_{\nu} + T_{\nu}{}^{\rho}{}_{\mu} - T^{\rho}{}_{\mu\nu}) = \mathring{\Gamma}^{\rho}{}_{\nu\mu} + K^{\rho}{}_{\mu\nu}$.

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$$R^{\lambda}{}_{\mu\sigma\nu} = \mathring{R}^{\lambda}{}_{\mu\sigma\nu} + \mathring{\nabla}_{\sigma}K_{\nu}{}^{\lambda}{}_{\mu} - \mathring{\nabla}_{\nu}K_{\sigma}{}^{\lambda}{}_{\mu} + K_{\sigma}{}^{\lambda}{}_{\rho}K_{\nu}{}^{\rho}{}_{\mu} - K_{\sigma}{}^{\rho}{}_{\mu}K_{\nu}{}^{\lambda}{}_{\rho} \equiv 0\,.$$

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- **Be careful here!** The general curvature $R^{\lambda}{}_{\mu\sigma\nu} \equiv 0$, not $\mathring{R}^{\lambda}{}_{\mu\sigma\nu} \neq 0$

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$$T = \frac{1}{4}T^{\rho}{}_{\mu\nu}T_{\rho}{}^{\mu\nu} + \frac{1}{2}T^{\rho}{}_{\mu\nu}T^{\nu\mu}{}_{\rho} - T^{\lambda}{}_{\lambda\mu}T_{\nu}{}^{\nu\mu}\,, \; B = \frac{2}{e}\partial_{\mu}(eT^{\lambda}{}_{\lambda}{}^{\mu})\,.$$

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- The Ricci scalar computed from the Levi-Civita connection $\mathring{R}$ differs from the scalar torsion T by a boundary term B.

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where $\kappa^2 = 8\pi G$ and $L_{\rm m}$ is any matter Lagrangian.

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Equivalence between field equations

The field equations arising from $S_{\rm TEGR}$ are equivalent to the Einstein field equations.

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- One interesting modified TG theory is when one considers[1]

[1] S. Bahamonde, C. G. Böhmer and M. Wright, Phys. Rev. D **92** (2015) no.10, 104042

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- There has been quite a lot of study about this theory in the last years.

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Field equations and spherical symmetry

- The separate tetrad and spin connection variations produce the field equations

$$W_{(\mu\nu)} = \kappa^2 \Theta_{\mu\nu}\,,$$
$$W_{[\mu\nu]} = \Big[(\partial_\rho f_B) + (\partial_\rho f_T)\Big] S_{[\mu}{}^{\rho}{}_{\nu]} \propto T^{\rho}{}_{[\mu\nu}\partial_{\rho]}(f_T + f_B) = 0\,.$$

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- In GR the relation among them corresponds to the Bianchi identity $\overset{\circ}{R}{}^{\alpha}{}_{\beta[\lambda\mu;\nu]} = 0$, which leads to covariant conservation of the Einstein tensor.
- We showed that for $f(T,B)$, if the antisymmetric part of equations is satisfied, then the covariant divergence of equations of motion vanishes identically.

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Killing eqs:

$$\mathcal{L}_{Z_\zeta} e^A{}_\mu = -\lambda_\zeta^A{}_B e^B{}_\mu\,, \quad \mathcal{L}_{Z_\zeta}\omega^A{}_{B\mu} = \partial_\mu \lambda_\zeta^A{}_B + \omega^A{}_{C\mu}\lambda_\zeta^C{}_B - \omega^C{}_{B\mu}\lambda_\zeta^A{}_C\,.$$

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By solving these eqs in the Weitzenböck gauge we find

$$e^A{}_\nu = \begin{pmatrix} C_1 & C_2 & 0 & 0 \\ C_3 \sin\vartheta\cos\varphi & C_4 \sin\vartheta\cos\varphi & C_5\cos\vartheta\cos\varphi - C_6\sin\varphi & -\sin\vartheta(C_5\sin\varphi + C_6\cos\vartheta\cos\varphi) \\ C_3\sin\vartheta\sin\varphi & C_4\sin\vartheta\sin\varphi & C_5\cos\vartheta\sin\varphi + C_6\cos\varphi & \sin\vartheta(C_5\cos\varphi - C_6\cos\vartheta\sin\varphi) \\ C_3\cos\vartheta & C_4\cos\vartheta & -C_5\sin\vartheta & C_6\sin^2\vartheta \end{pmatrix},$$

where the six free functions $C_I = C_I(t,r)$ ($I = 1, ..6$) can depend on time and the radial coordinate

Solving antisymmetric field equations

There are two different tetrads which solve the antisymmetric field equation and they have the same metric[2]

$$e^{A}_{(1)\,\mu} = \begin{pmatrix} \mathcal{A}(r) & 0 & 0 & 0 \\ 0 & \mathcal{B}(r)\sin\vartheta\cos\varphi & \xi r\cos\vartheta\cos\varphi & -r\xi\sin\vartheta\sin\varphi \\ 0 & \mathcal{B}(r)\sin\vartheta\sin\varphi & \xi r\cos\vartheta\sin\varphi & \xi r\sin\vartheta\cos\varphi \\ 0 & \mathcal{B}(r)\cos\vartheta & -r\xi\sin\vartheta & 0 \end{pmatrix}, \quad \xi = \pm 1\,,$$

$$e^{A}_{(2)\,\mu} = \begin{pmatrix} 0 & i\mathcal{B}(r) & 0 & 0 \\ i\mathcal{A}(r)\sin\vartheta\cos\varphi & 0 & -r\sin\varphi & -r\sin\vartheta\cos\vartheta\cos\varphi \\ i\mathcal{A}(r)\sin\vartheta\sin\varphi & 0 & \chi r\cos\varphi & -r\sin\vartheta\cos\vartheta\sin\varphi \\ i\mathcal{A}(r)\cos\vartheta & 0 & 0 & r\sin^2\vartheta \end{pmatrix},$$

$$\mathrm{d}s^2 = \mathcal{A}(r)^2\mathrm{d}t^2 - \mathcal{B}(r)^2\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2\,.$$

[2] S. Bahamonde, A. Golovnev, M. J. Guzmán, J. L. Said and C. Pfeifer, JCAP **01** (2022) no.01, 037

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The phenomenology of these two tetrads will be different!

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- $e_{(1)\mu}^{A}$ has been derived before but different papers did not notice that $\xi=+1$ has a different value of T, B than $\xi=-1$ and then different predictions.

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- For the complex tetrad, all the quantities (torsion tensor, scalars, etc) are real.
- Since we couple matter with the metric, these imaginary terms are not seen, so nothing is wrong with it.

Solutions for the complex tetrad - similar RN

- The first interesting black hole one is

$$\mathrm{d}s^2 = \Big(1 - \frac{2M}{r} + \frac{Q}{r^2}\Big)dt^2 - \Big(\frac{2Mr - Q - r^2}{2Q - r^2}\Big)^{-1}dr^2 - r^2 d\Omega^2\,,$$

which looks like Reissner–Nordström but it does not have $g_{tt} = -1/g_{rr}$.

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- This solution has two event horizons and can have any sign for Q.
- The form of the theory is

$$f(T) = 4f_0\,\frac{\Big(2 \pm \sqrt{Q^2T^2 - 2QT + 4}\Big)}{\Big(QT + 2 \pm \sqrt{Q^2T^2 - 2QT + 4}\Big)\sqrt{8 - 2QT \pm 4\sqrt{Q^2T^2 - 2QT + 4}}}\,.$$

Solutions for the complex tetrad - Born-Infield

- A quite well-studied theory inspired by Born-Infield electromagnetism is

$$f(T) = \lambda\left(\sqrt{1+\frac{2T}{\lambda}}-1\right),$$

with λ being the so-called Born-Infeld parameter. It is easy to notice that when $T/\lambda \ll 1$, one obtains $f(T) = T - T^2/(2\lambda) + \mathcal{O}(1/\lambda^2)$.

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- We found an exact black hole solution to this theory

$$\begin{aligned}\mathrm{d}s^2 &= \frac{a_1^2}{r}\Big[\sqrt{\lambda}(a_0\lambda+r)-2\tan^{-1}\left(\frac{\sqrt{\lambda}r}{2}\right)\Big]\mathrm{d}t^2\\ &\quad-\frac{\lambda^{5/2}r^5}{(4+r^2\lambda)^2}\Big[\sqrt{\lambda}(a_0\lambda+r)-2\tan^{-1}\left(\frac{\sqrt{\lambda}r}{2}\right)\Big]^{-1}\mathrm{d}r^2-r^2\mathrm{d}\Omega^2\,.\end{aligned}$$

Solutions for the complex tetrad - Born-Infield

- One can set $a_1^2 = 1/\sqrt{\lambda}$ to get asymptotically flatness. Further, if we choose $a_0 = -2M/\lambda$ and expands the metric up to $\mathcal{O}(1/\lambda^2)$, we find

$$ds^2 = \Big[1 - \frac{2M}{r} + \frac{4}{\lambda r^2} - \frac{\pi}{\sqrt{\lambda} r}\Big] dt^2 - \Big[1 - \frac{2M}{r} - \frac{16M}{\lambda r^3} + \frac{12}{\lambda r^2} - \frac{\pi}{\sqrt{\lambda} r}\Big]^{-1} dr^2 - r^2 d\Omega^2 + \mathcal{O}(1/\lambda^2)\,.$$

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- This is a generalization of a Schwarzschild black hole with one horizon $r_h = 2M + \frac{\pi}{\sqrt{\lambda}} - \frac{2M}{\lambda} + \mathcal{O}(1/\lambda^2)$.

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$$S = \frac{1}{2\kappa^2}\int_M \left[\mathcal{F}(\psi)\mathring{R} + 2\mathcal{B}(\psi)X - 2\kappa^2\mathcal{V}(\psi)\right]\sqrt{-g}\,\mathrm{d}^4x\,.$$

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- One can circumvent the no-hair theorem by having some particular potentials and coupling functions.
- It is well known that spontenous scalarization of black holes occur in other theories like the ones with coupling with Gauss-Bonnet. (see Ludovic's talk)

Scalar-torsion theories

- One can formulate Teleparallel theories with a scalar field, for example[3]:

$$S=\frac{1}{2\kappa^2}\int_M\left[-\mathcal{A}(\psi)T-\tilde{\mathcal{C}}(\psi)B+2\mathcal{B}(\psi)X-2\kappa^2\mathcal{V}(\psi)\right]\sqrt{-g}\,\mathrm{d}^4x\,,$$

where $X=-\frac{1}{2}g^{\mu\nu}\partial_\mu\psi\partial_\nu\psi$.

[3] S. Bahamonde and M. Wright, Phys. Rev. D **92** (2015) no.8, 084034; M. Zubair, S. Bahamonde and M. Jamil, Eur. Phys. J. C **77** (2017) no.7, 472; M. Hohmann and C. Pfeifer, Phys. Rev. D **98** (2018) no.6, 064003.

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  where $X = -\frac{1}{2}g^{\mu\nu}\partial_\mu\psi\partial_\nu\psi$.

- Since $\mathring{R} = -T + B$, when $\mathcal{A}(\psi) = -\tilde{\mathcal{C}}(\psi)$ the above theory is exactly the same as the standard non-minimally one..

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- The Eqs can be written as

$$\mathring{R}_{\mu\nu} - \frac{1}{2}\mathring{R}g_{\mu\nu} = -\frac{1}{\mathcal{A}(\psi)}\Big[\left(\mathcal{A}'(\psi) + \tilde{\mathcal{C}}'(\psi)\right) S_{(\mu\nu)}{}^{\rho}\psi_{,\rho} + \left(\frac{1}{2}\beta - \tilde{\mathcal{C}}''(\psi)\right)\psi_{,\rho}\psi_{,\sigma}g^{\rho\sigma}g_{\mu\nu}$$
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- By solving the antisymmetric eqs, one gets the same two possible tetrads presented before, with the metric now as

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- We can express the energy of the scalar field in the black hole exterior region as

$$\mathcal{E}_{\psi} = \int_{r_h}^{\infty}\int_{S^2}\Theta^{(\psi)t}{}_{t}Cr^2\mathrm{d}r\,\mathrm{d}\Omega = 4\pi\int_{r_h}^{\infty}\rho_{\psi}(r)Cr^2\mathrm{d}r\,,$$

where $\rho_{\psi}(r) = \Theta^{(\psi)t}{}_{t}(r)$ will be interpreted as the (effective) scalar field energy density.

$\mathcal{A}(\psi) = \alpha$ only non-minimal coupling between the scalar field and the boundary

- This theory has a GR term plus and a $\tilde{\mathcal{C}}(\psi)B$ term.

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- For the real tetrad we only found solutions that do not look interesting[4].
- For the complex tetrad, we found Schwarzschild-de-Sitter solutions with different couplings and potentials.
- We also found a non-asymptotically flat solution

$$\begin{aligned}\mathrm{d}s^2 &= r^{-2}\Big(A_0 r^{\pm\sqrt{2w}}+r^{\pm\sqrt{w/2}}\Big)\mathrm{d}t^2\\ &\quad -\frac{A_0\sqrt{w}}{2}r^{\pm\sqrt{w/2}}\Big(A_0 r^{\pm\sqrt{w/2}}+1\Big)^{-1}(\pm2\sqrt{2}-\sqrt{w})\mathrm{d}r^2-r^2\mathrm{d}\Omega^2\,,\end{aligned}$$

with $w=2-\beta\psi_0^2$ and

$$\tilde{\mathcal{C}}(\psi)=\Big(\frac{-2\pm\sqrt{2w}}{\psi_0}\Big)\psi\,,\quad \mathcal{V}(\psi)=0\,,\quad \psi=\psi_0\log(r)\,.$$

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- One interesting one is

$$\mathrm{d}s^2 = \left(1 - \frac{K}{r}\right)^2 \mathrm{d}t^2 - \left(1 - \frac{K}{r}\right)^{-2} \mathrm{d}r^2 - r^2 \mathrm{d}\Omega^2 \,,$$

with $\psi(r) = -\frac{2\psi_0\sqrt{r}}{K\sqrt{r-K}}, \mathcal{A}(\psi) = -\frac{1}{8}\beta\psi^2 \,, \mathcal{V}(\psi) = 0$.

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with $\psi(r)=-\frac{2\psi_0\sqrt{r}}{K\sqrt{r-K}},\mathcal{A}(\psi)=-\frac{1}{8}\beta\psi^2\,,\mathcal{V}(\psi)=0$.

- The metric is the same as the Bocharova–Bronnikov–Melnikov–Bekenstein (BBMB) solution found in Riemannian conformal scalar-vacuum theory!

$\tilde{\mathcal{C}}(\psi)=0$ only non-minimal coupling between the scalar field and the torsion scalar

- Another interesting scalarised black hole solution found is

$$\begin{aligned} \mathrm{d}s^2 &= \Big(2-\frac{r}{2K}+\frac{\sqrt{r(r-4K)}}{2K}\Big)^2\mathrm{d}t^2 \\ &\quad -\Big(2-\frac{r}{2K}+\frac{\sqrt{r(r-4K)}}{2K}\Big)^{-2}\mathrm{d}r^2-r^2\mathrm{d}\Omega^2\,, \end{aligned}$$

where the scalar field is non-trivial and $\mathcal{A}(\psi)=\frac{3\beta}{8}\psi^2$. This metric is asymptotically flat.

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- If $K\ll 1$ the metric components behave as

$$g_{tt}=-1/g_{rr}=1-\frac{2K}{r}-\frac{3K^2}{r^2}+\mathcal{O}(K^3)\,,$$

which looks like a Reissner-Norstrdom BH with imaginary charge ($Q^2=-3M^2$).

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- Another exact BH solution (non-asymptotically flat) that we found is

$$\mathrm{d}s^2 = \Big(A_0 - 2Mr^{-2p-1}\Big)\mathrm{d}t^2 - \Big(\frac{A_0(2p+1)}{A_0-2Mr^{-2p-1}}\Big)\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2\,,$$

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$$\mathcal{A}(\psi) = -\frac{1}{4}p\beta\psi^2\,, \quad \psi(r) = \psi_0 + p\phi_0\log(r) + \mathcal{O}(p^2)\,, \quad \mathcal{V}(\psi) = 0\,, \quad p \neq 0\,.$$

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- This solution then corresponds to a scalarised black hole solution with p being the parameter that controls the scalar field.
- If $p \ll 1$ we find

$$\begin{aligned}\mathrm{d}s^2 &= \Big[1 - \frac{2M}{r} + \frac{4Mp}{r}\log(r)\Big]\mathrm{d}t^2 - \Big(1-\frac{2M}{r}\Big)^{-1}\Big[1 + 2p - \Big(1-\frac{2M}{r}\Big)^{-1}\frac{4Mp}{r}\log(r)\Big]\mathrm{d}r^2\\ &\quad - r^2\mathrm{d}\Omega^2 + \mathcal{O}(p^2)\,,\end{aligned}$$

$$\mathcal{A}(\psi) = -\frac{1}{4}p\beta\psi^2\,,\quad \psi(r) = \psi_0 + p\phi_0\log(r) + \mathcal{O}(p^2)\,,\quad \mathcal{V}(\psi) = 0\,,\quad p \neq 0\,.$$

- This expanded form of the metric contains a logarithmic term as in $f(T)$

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4. $\mathcal{A}=\alpha$, $\tilde{\mathcal{C}}=\frac{\gamma}{m+1}\psi^{m+1}$ and $\frac{1}{\beta}\left(\psi\mathcal{V}'-(m+1)\gamma\psi^m T^r\psi'\right)\leq 0$ or $\frac{(m+1)}{m-1}\frac{1}{\beta}\left(\alpha\mathring{R}+\kappa^2(\psi\mathcal{V}'-4\mathcal{V})\right)\leq 0$.

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- Are these solutions stable? what is the phenomenology of them?