New types of hairy Black Holes sourced by Torsion

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IBS seminar, Korea, Dajeon; 1/August/2023

JCAP **01** (2022) no.01, 037; Phys. Rev. D **106** (2022) no.8, 084046; Eur. Phys. J. C **82** (2022) no.11, 1018; Phys. Rev. D **107** (2023) no.10, 104013; and arXiv:2307.14720 (last week)+arXiv:2308.1XXX +arXiv:2308.2XXX

東京工業大学
Tokyo Institute of Technology

Overview of the Talk

Fundamental variables and characteristic tensors

- In the most general metric-affine setting, the fundamental variables are a **metric** $g_{\mu\nu}$ (10 comp.) as well as the coefficients $\hat{\Gamma}^{\rho}{}_{\mu\nu}$ (64 comp.) of an **affine connection**.

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Torsion	$\tilde{T}^{\mu}{}_{\nu\rho} = \tilde{\Gamma}^{\mu}{}_{\rho\nu} - \tilde{\Gamma}^{\mu}{}_{\nu\rho}$
Nonmetricity	$\tilde{Q}_{\mu\nu\rho} = \tilde{\nabla}_{\mu}g_{\nu\rho} = \partial_{\mu}g_{\nu\rho} - \tilde{\Gamma}^{\sigma}{}_{\nu\mu}g_{\sigma\rho} - \tilde{\Gamma}^{\sigma}{}_{\rho\mu}g_{\nu\sigma}$

Tetrads and spin connection

- **Notation:** $\mu, \nu, \alpha, ..$: space-time; $a, b, c, A, B, C, ..$: tangent space. $\mathring{\Gamma}$: Levi-Civita, Γ: Teleparallel connection;$\tilde{\Gamma}$: General connection.

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where η_{ab} is the Minkowski metric.

- Quantities denoted with a circle on top $\circ$ denote that they are defined with respect to the Levi-Civita connection and hats are general affine connection.

Curvature tensor

- The curvature becomes

$$\tilde{R}^{\mu}{}_{\nu\rho\sigma} = \mathring{R}^{\mu}{}_{\nu\rho\sigma} + \mathring{\nabla}_{\rho}\tilde{D}^{\mu}{}_{\nu\sigma} - \mathring{\nabla}_{\sigma}\tilde{D}^{\mu}{}_{\nu\rho} + \tilde{D}^{\mu}{}_{\tau\rho}\tilde{D}^{\tau}{}_{\nu\sigma} - \tilde{D}^{\mu}{}_{\tau\sigma}\tilde{D}^{\tau}{}_{\nu\rho}\,.$$

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$$\tilde{R} = \mathring{R} + \Big(T + 2\mathring{\nabla}_{\mu}(\sqrt{-g}T^{\rho}{}_{\rho}{}^{\mu})\Big) + \Big(Q + \mathring{\nabla}_{\mu}Q^{\mu\nu}{}_{\nu} - \mathring{\nabla}_{\nu}Q_{\mu}{}^{\mu\nu}\Big) + C$$

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with

$$T := T^{\rho\lambda\kappa}T_{\rho\lambda\kappa} + 2T^{\rho\lambda\kappa}T_{\kappa\rho\lambda} - 4T_{\rho}{}^{\kappa}{}_{\kappa}T^{\rho\lambda}{}_{\lambda}\,, \quad \text{Torsion scalar}\,,$$

$$Q := -\frac{1}{4}Q_{\alpha\beta\gamma}Q^{\alpha\beta\gamma} + \frac{1}{2}Q_{\alpha\beta\gamma}Q^{\beta\alpha\gamma} + \frac{1}{4}Q_{\alpha}Q^{\alpha} - \frac{1}{2}Q_{\alpha}\bar{Q}^{\alpha}\,, \ \text{Nonmetricity scalar}\,,$$

$$C := 2(Q_{\kappa\rho\lambda}T^{\lambda\kappa\rho} + Q_{\rho}{}^{\sigma}{}_{\sigma}T^{\rho\kappa}{}_{\kappa} - Q^{\sigma}{}_{\sigma\rho}T^{\rho\kappa}{}_{\kappa})\,.$$

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- Since $\mathring{R}$ differs by T by a boundary term B, **the equations of TEGR are equivalent to the Einstein's field equations.**

Overview of the Talk

Connection in Teleparallel gravity

- GR assumes zero torsion and non-zero curvature (Levi-Civita connection) whereas Teleparallel gravity (TG) assumes **zero curvature and non-zero torsion.** Both assumes non-metricity to be zero, meaning $\tilde{\nabla}_\alpha g_{\mu\nu} = 0$.

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- In TG, it is always possible to find a frame such that $\omega^a{}_{b\mu} = 0$, but this is a gauge choice, so only some tetrads are compatible with this[1].

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- The torsion tensor is generally non-vanishing, and transforms covariantly under both diffeomorphisms and local Lorentz transformations.
- The pure tetrad formalism was the initial framework used for TG, which chooses a specific frame where the spin connection $\omega^{a}{}_{b\mu}$ vanishes. Be careful choosing the correct tetrad which is compatible with this gauge.

Important properties of Teleparallel theories

- Teleparallel theories have the tetrads and spin connection as the fundamental variables, so that, one most commonly assumes an action which is of the form

$$\mathcal{S} = \mathcal{S}_{\mathsf{g}}[e, \omega] + \mathcal{S}_{\mathsf{m}}[e, \chi]\,,$$

where the gravitational part $\mathcal{S}_{\mathsf{g}}$ of the action depends on the tetrad $e^A{}_\mu$ and the spin connection $\omega^A{}_{B\mu}$, while the matter part depends on the tetrad $e^A{}_\mu$ and arbitrary matter fields χ^I, but not on the spin connection.

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- Particles (bosonic or femionic) follow the standard geodesic equation.

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- Variations of any action should be taken with respect to both variables (with the emphasis that the spin connection is non-arbitrary but always flat).

[2] M. Hohmann, L. Järv, M. Krššák and C. Pfeifer, Phys. Rev. D **97** (2018) no.10, 104042

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- Since $\omega^{A}{}_{B\mu}$ is a pure-gauge quantity, it can be shown that the the antisymmetric part of the field equations arising from variations w/r to the tetrads $e^{A}{}_{\mu}$ coincides with the variations of the action w/r to $\omega^{A}{}_{B\mu}$.

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- Then, in the Weitzenbock gauge (zero spin connection), it is sufficient to: [2]

$$\delta_e S \Longrightarrow E_{(\mu\nu)} = \kappa^2 \Theta_{\mu\nu}\,, \quad E_{[\mu\nu]} = 0\,, \tag{2}$$

then, we have $10+6$ dof of the tetrad in the symmetric+antisymmetric field equations.

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Spherical Symmetry in Teleparallel gravity

- In TG, all the dynamics can be put in the tetrad and the spin connection can be set to be zero.

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- We assume that the connection and metric have the same symmetries:

$$\mathcal{L}_{Z_\zeta} e^A{}_\mu = -\lambda_\zeta^A{}_B e^B{}_\mu \,, \quad \mathcal{L}_{Z_\zeta} \omega^A{}_{B\mu} = \partial_\mu \lambda_\zeta^A{}_B + \omega^A{}_{C\mu} \lambda_\zeta^C{}_B - \omega^C{}_{B\mu} \lambda_\zeta^A{}_C \,. \tag{3}$$

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- The most general tetrad satisfying spherical symmetry in the Weitzenbock gauge (zero spin connection) is [3]

$$e^A{}_\nu = \begin{pmatrix} C_1 & C_2 & 0 & 0 \\ C_3 \sin\theta\cos\phi & C_4 \sin\theta\cos\phi & C_5\cos\theta\cos\phi - C_6\sin\phi & -\sin\theta(C_5\sin\phi + C_6\cos\theta\cos\phi) \\ C_3\sin\theta\sin\phi & C_4\sin\theta\sin\phi & C_5\cos\theta\sin\phi + C_6\cos\phi & \sin\theta(C_5\cos\phi - C_6\cos\theta\sin\phi) \\ C_3\cos\theta & C_4\cos\theta & -C_5\sin\theta & C_6\sin^2\theta \end{pmatrix},$$

where $C_i = C_i(t,r)$.

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- The most general tetrad satisfying spherical symmetry in the Weitzenbock gauge (zero spin connection) is [3]

$$e^A{}_\nu = \begin{pmatrix} C_1 & C_2 & 0 & 0 \\ C_3 \sin\theta\cos\phi & C_4 \sin\theta\cos\phi & C_5\cos\theta\cos\phi - C_6\sin\phi & -\sin\theta(C_5\sin\phi + C_6\cos\theta\cos\phi) \\ C_3\sin\theta\sin\phi & C_4\sin\theta\sin\phi & C_5\cos\theta\sin\phi + C_6\cos\phi & \sin\theta(C_5\cos\phi - C_6\cos\theta\sin\phi) \\ C_3\cos\theta & C_4\cos\theta & -C_5\sin\theta & C_6\sin^2\theta \end{pmatrix},$$

where $C_i = C_i(t,r)$.

- Using $g_{\mu\nu} = \eta_{AB} e^A{}_\mu e^B{}_\nu$, we have that the metric is

$$\begin{aligned} \mathrm{d}s^2 \;&=\; \left(C_1^2 - C_3^2\right)\mathrm{d}t^2 - 2(C_3C_4 - C_1C_2)\,\mathrm{d}t\,\mathrm{d}r - \left(C_4^2 - C_2^2\right)\mathrm{d}r^2 \\ &\quad - \left(C_5^2 + C_6^2\right)\left(\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2\right), \end{aligned}$$

where we have cross-terms.

[3] M. Hohmann, L. Järv, M. Krššák and C. Pfeifer, Phys. Rev. D **100** (2019) no.8, 084002

Spherical Symmetry in Teleparallel gravity

- Without losing generality, we can choose a coordinate system such that the cross term vanishes. This can be easily done by taking the following reparametrization:

$$\begin{aligned} C_1(r) &= \nu A(r)\cosh\beta(r)\,, \quad C_3(r) = \nu A(r)\sinh\beta(r)\,, \\ C_4(r) &= \xi B(r)\cosh\beta(r)\,, \quad C_2(r) = \xi B(r)\sinh\beta(r)\,, \\ C_5(r) &= \chi C(r)\cos\alpha(r)\,, \quad C_6(r) = \chi C(r)\sin\alpha(r)\,, \end{aligned}$$

  with $\{\nu,\xi,\chi\}$ being ±1. This tetrad gives the metric in the standard form in spherical coordinates:

$$ds^2 = A(r)^2\,dt^2 - B(r)^2\,dr^2 - C(r)^2(d\theta^2 + \sin^2\theta d\phi^2)\,.$$

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- Note that $\beta(r), \alpha(r)$ are tetrad dof (they do not appear in the metric). They can be set by solving the antisymmetric field equations.

Overview of the Talk

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- This is a theorem, meaning that in GR, this is the unique asymptotically flat black hole solution. In spherical symmetry, it is just Schwarzschild.
- Is it possible that black holes have hair? One needs to go beyond GR, either by having modified gravity or allowing extra degrees of freedom such as scalar fields coupled to gravity.

$f(T)$ gravity

- Inspired from $f(\mathring{R})$ gravity, Ferraro and Fiorini[4] introduced another teleparallel theory by generalising $T \to f(T)$ in the action:

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- The second branch branch which solves the antisymmetric equations is $\beta(r) = \frac{i\pi}{2} + i\pi\, n_3\,, \alpha(r) = \frac{\pi}{2} + \pi$ which gives

$$e^{(2)a}{}_{\mu} = \begin{pmatrix} 0 & i\xi B & 0 & 0 \\ i\nu A\sin\theta\cos\phi & 0 & -\chi C\sin\phi & -\chi C\sin\theta\cos\theta\cos\phi \\ i\nu A\sin\theta\sin\phi & 0 & \chi C\cos\phi & -\chi C\sin\theta\cos\theta\sin\phi \\ i\nu A\cos\theta & 0 & 0 & \chi C\sin^2\theta \end{pmatrix}, \quad \{\nu,\xi,\chi\} = \pm 1\,.$$

Born-Infield $f(T)$ Black-Hole Solution

- A quite well-studied theory inspired by Born-Infield electromagnetism is

$$f(T) = \lambda\left(\sqrt{1+\frac{2T}{\lambda}}-1\right),$$

with λ being the so-called Born-Infeld parameter. It is easy to notice that when $T/\lambda \ll 1$, one obtains $f(T) = T - T^2/(2\lambda) + \mathcal{O}(1/\lambda^2)$.

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- We found an exact black hole solution to this theory [5]

$$\begin{aligned}\mathrm{d}s^2 = {} & \frac{a_1^2}{r}\Big[\sqrt{\lambda}(a_0\lambda + r) - 2\tan^{-1}\left(\frac{\sqrt{\lambda}r}{2}\right)\Big]\mathrm{d}t^2 \\ & - \frac{\lambda^{5/2}r^5}{(4+r^2\lambda)^2}\Big[\sqrt{\lambda}(a_0\lambda + r) - 2\tan^{-1}\left(\frac{\sqrt{\lambda}r}{2}\right)\Big]^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2\,.\end{aligned}$$

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- One can set $a_1^2 = 1/\sqrt{\lambda}$ to get asymptotically flatness. Further, if we choose $a_0 = -2M/\lambda$ and expands the metric up to $\mathcal{O}(1/\lambda^2)$, we find

$$ds^2 = \Big[1 - \frac{2M}{r} + \frac{4}{\lambda r^2} - \frac{\pi}{\sqrt{\lambda} r}\Big] dt^2 - \Big[1 - \frac{2M}{r} - \frac{16M}{\lambda r^3} + \frac{12}{\lambda r^2} - \frac{\pi}{\sqrt{\lambda} r}\Big]^{-1} dr^2 - r^2 d\Omega^2 + \mathcal{O}(1/\lambda^2)\,.$$

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- This is a generalization of a Schwarzschild black hole with one horizon $r_h = 2M + \frac{\pi}{\sqrt{\lambda}} - \frac{2M}{\lambda} + \mathcal{O}(1/\lambda^2)$.

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- It is also useful to calculate the Komar mass $\mathcal{M}$ for this spacetime which is related to the force needed by an observer at infinity to keep a spherical uniform mass distribution.

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- For that, one must assume the existence of a timelike Killing vector field $^{\mu} = \{1, 0, 0, 0\}$ and a spacelike hypersurface Σ_t from the event horizon to spatial infinity in a constant slice t whose normal vector is $n_\mu = \{-\sqrt{g_{tt}}, 0, 0, 0\}$. Then, the Komar mass reads

$$\mathcal{M} = -\frac{1}{8\pi}\int_{S_t} \overset{\circ}{\nabla}{}^{\mu} dS_\mu \,, \tag{4}$$

where S_t is the 2-boundary of Σ_t and dS_μ is the surface element of S_t which is $dS_\mu = -2n_{[\mu}\sigma_{]}\sqrt{h}d\theta d\phi$ with $h = r^2\sin\theta$ being the determinant of the 2-dimensional metric on S_t and $\sigma_\mu = \{0, \sqrt{g_{rr}}, 0, 0\}$.

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- For our solution we find [6]

$$\mathcal{M} = \frac{1}{8\pi}\lim_{r\to\infty}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\frac{r^2 g'_{tt}}{\sqrt{g_{tt}g_{rr}}}\sin\theta = \left(\frac{\pi}{2\lambda}+1\right)M \,. \tag{5}$$

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- Expanding the metric for large $\frac{r}{M}$ to study the weak field limit gives

$$-g_{tt} = 1 - \frac{2M}{r}\left(1 + \frac{\pi}{2\lambda}\right) + \frac{4M^2}{r^2\lambda^2} + \mathcal{O}(r^{-4})\,, \tag{6}$$

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where $\hat{M} = Gm/c^2$ and m is the Newtonian mass, we find consistently that $\hat{M} = \mathcal{M}$.

- Thus, $\gamma = 1$ agrees with its GR value; however

$$(\beta - 1) = \frac{8}{\pi^2}\left(1 - \frac{M}{\hat{M}}\right)^2 = \frac{8}{(2\lambda + \pi)^2}, \quad (10)$$

attains a correction. Using the observational bound that $|\beta - 1| < 10^{-4}$, we find $\lambda \gtrsim 140$.

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- Hence the influence of rotation on the accretion disk is way larger than the influence of the teleparallel parameter.

Born-Infield $f(T)$ Black-Hole Solution - Thick accretion disks

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- Moreover, this model is axisymmetric and stationary:

$$u^\mu = (u^t, 0, 0, u^\phi)\,, \tag{11}$$

$$T^\mu{}_\nu = (\epsilon + p)u_\nu u^\mu + \delta^\mu{}_\nu p\,, \tag{12}$$

where $u_t = \sqrt{\frac{g_{tt}g_{\phi\phi}}{l^2 g_{tt} + g_{\phi\phi}}}$ and l is a constant of motion $l = \frac{L}{E}$; $\Omega = \frac{u^\phi}{u^t}$ is the angular velocity.

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- W provides the equipotential surfaces topology of the disk.

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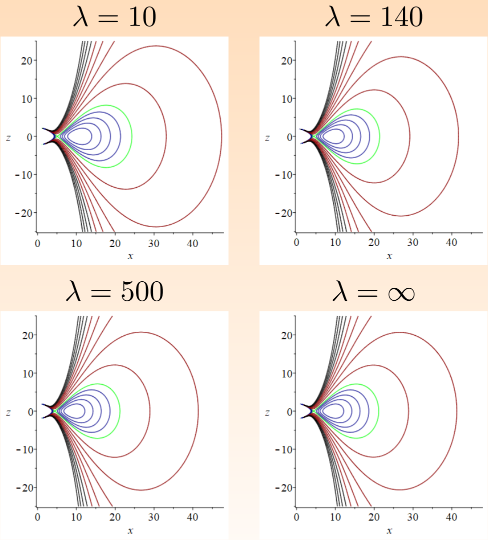
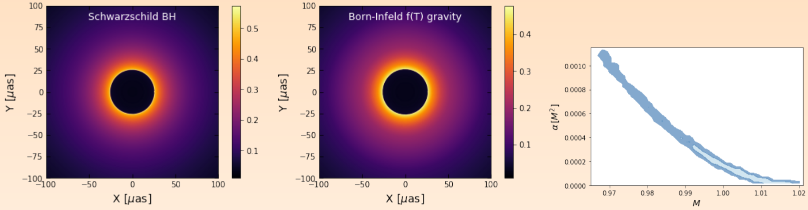


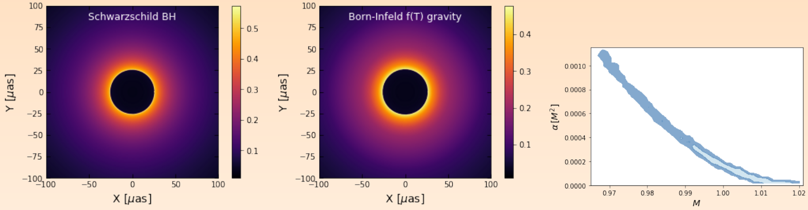
Figure: Equipotential surfaces for different choices of λ and constant angular momentum. Green line:torus with a cusp; Blue lines indicate closed tori; red lines bound structures without inner edge; and black lines open surfaces.



- **Born-Infeld $f(T)$ black hole solution consistent with S2 star observations** after a Monte-Carlo-Markov Chains analysis. [7]

[7] K. Jusufi, S. Capozziello, S. Bahamonde and M. Jamil, Eur. Phys. J. C **82** (2022) no.11, 1018

Born-Infield $f(T)$ Black-Hole Solution - S2 and Shadow



- **Born-Infeld $f(T)$ black hole solution consistent with S2 star observations** after a Monte-Carlo-Markov Chains analysis. [7]
- **Shadow images of Sgr A* black hole:** Angular diameter of the Born-Infeld $f(T)$ gravity is consistent with the observation and could not be distinguished from the Schwarzschild black hole in GR by the present technology.

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Overview of the Talk

New General Relativity (NGR)

The torsion tensor can be decomposed in its irreducible parts as

$$a_\mu = \frac{1}{6}\epsilon_{\mu\nu\sigma\rho}T^{\nu\sigma\rho}\,, \quad v_\mu = T^{\sigma}{}_{\sigma\mu}\,,$$
$$t_{\sigma\mu\nu} = \frac{1}{2}\left(T_{\sigma\mu\nu} + T_{\mu\sigma\nu}\right) + \frac{1}{6}\left(g_{\nu\sigma}v_\mu + g_{\nu\mu}v_\sigma\right) - \frac{1}{3}g_{\sigma\mu}v_\nu\,,$$

where $\epsilon_{\mu\nu\sigma\rho}$ is the totally anti-symmetric Levi-Civita symbol.
From these we build the scalars

$$T_{\text{ax}} = a_\mu a^\mu\,, \quad T_{\text{vec}} = v_\mu v^\mu\,, \quad T_{\text{ten}} = t_{\sigma\mu\nu}t^{\sigma\mu\nu}\,,$$

and the torsion scalar is a linear combination

$$T = \frac{3}{2}T_{\text{ax}} + \frac{2}{3}T_{\text{ten}} - \frac{2}{3}T_{\text{vec}}\,.$$

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$$\mathcal{S}_{\rm NGR} = \frac{1}{2\kappa^2}\int d^4x \Big[c_1 T_{\rm vec} + c_2 T_{\rm ax} + c_3 T_{\rm ten}\Big] e\,.$$

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- If $c_1 = -\frac{2}{3}$, $c_2 = \frac{3}{2}$, $c_3 = \frac{2}{3}$, the above action is equivalent to the TEGR one.

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New General Relativity (NGR)

- The theory has ghosts unless $c_{\rm ten}+c_{\rm vec}=0$, which gives the so-called 1-parameter New GR theory:

$$S_{\rm 1NGR}=\frac{1}{2\kappa^2}\int d^4xe(T+\epsilon\,T_{\rm axi})\,, \tag{14}$$

where

$$c_{\rm vec}=-\frac{2}{3}\,,\quad c_{\rm ten}=\frac{2}{3}\,,\quad c_{\rm axi}=\frac{3}{2}+\epsilon\,. \tag{15}$$

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- This modification might be indistinguishable from GR, or not?

- The antisymmetric field equation can be solved in 3 different ways giving three branches. Recall $\alpha(t,r), \beta(t,r)$ are the tetrad extra dof. (Bahamonde et. al arXiv:2308.1XXX)

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- **Second Branch:** We found an exact Black hole solution given by:

$$ds^2 = \Big(1 - \frac{2M}{r}\Big)dt^2 - \Big(1 - \frac{2M}{r}\Big)^{-1} h(\alpha,\epsilon)dr^2 - r^2 d\Omega^2\,, \tag{16}$$

where

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- Note that the Komar mass is

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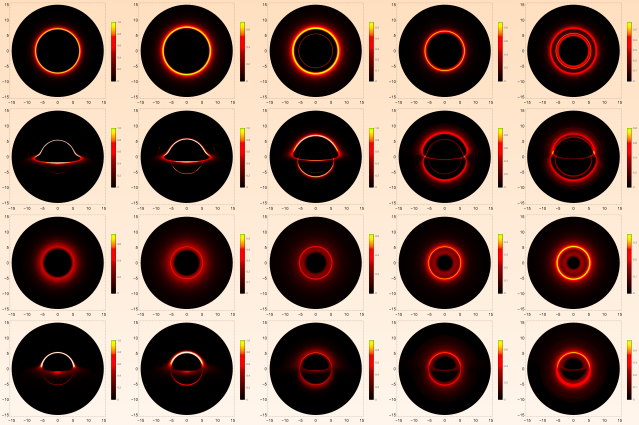

Figure: Observed images for the ISCO disk model (top two rows) and the EH disk model (bottom row rows) with inclination angle of 0° (top) and 80° (bottom), for $\bar{\delta} = \{-2; -1; 0; 0.5; 0.7\}$, from left to right ($h = \frac{1}{1-\bar{\delta}}$).

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$$\mathring{G} = \mathring{R}_{\alpha\beta\mu\nu}\mathring{R}^{\alpha\beta\mu\nu} - 4\mathring{R}_{\alpha\beta}\mathring{R}^{\alpha\beta} + \mathring{R}^2\,.$$

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  gives the Einstein's field equations.
- However, if one allows couplings between the Gauss-Bonnet invariant and a scalar field, then the field equations will not be longer equivalent to GR.

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- The Scalar Gauss-Bonnet (sGB) gravity theory is described by the following action

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  where $\dot{\mathcal{G}}(\psi) = d\mathcal{G}/d\psi$.
- Note: $\mathring{R} = 0$ in Schwarzschild but $\mathring{G} \neq 0$. That property would be important to understand the difference between this model and the former one.

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- This transition is usually smooth in sGB and shares similarities with second order phase transitions.

Scalar fields non-minimally coupled to Torsion

- In our previous paper[9], we studied Teleparallel theories with a scalar field, for example[10]:

$$S = \frac{1}{2\kappa^2}\int_M \left[-\mathcal{A}(\psi)T - \tilde{\mathcal{C}}(\psi)B - \frac{1}{2}\mathcal{B}(\psi)\partial_\mu\psi\partial^\mu\psi - 2\kappa^2\mathcal{V}(\psi)\right]\sqrt{-g}\,\mathrm{d}^4x\,,$$

where $X = -\frac{1}{2}g^{\mu\nu}\partial_\mu\psi\partial_\nu\psi$.

[9] S. Bahamonde, L. Ducobu and C. Pfeifer, JCAP **04** (2022) no.04, 018

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- Since $\mathring{R} = -T + B$, when $\mathcal{A}(\psi) = -\tilde{\mathcal{C}}(\psi)$ the above theory is exactly the same as the standard non-minimally one.
- We found new exact black hole solutions (some of them different to the Riemannian case), but they seem to be not so much interesting phenomenologically.

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- It is convenient to re-parametrize the action such that one has the Riemannian case: (Note again $\mathring{G} = T_G + B_G$)

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- There are three important limiting cases appearing from the above action:
  1. $\alpha_3 = 0$ (or equivalently $\alpha_1\mathcal{G}_1(\psi) = \alpha_2\mathcal{G}_2(\psi)$): this theory corresponds to the standard sGB theory.
  2. $\alpha_2 = 0$: this theory corresponds to a purely Teleparallel theory where the dynamics are governed by $F(\psi)T_G$.

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$$\mathcal{S}_{\rm TsGB} = \frac{1}{2\kappa^2}\int \Big[-T - \frac{1}{2}\beta\,\partial_\mu\psi\partial^\mu\psi + \alpha_2\mathcal{G}_2(\psi)(\mathring{G} - T_G) + \alpha_1\mathcal{G}_1(\psi)T_G \Big]\, e\,\mathrm{d}^4x$$
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- There are three important limiting cases appearing from the above action:
  1. $\alpha_3 = 0$ (or equivalently $\alpha_1\mathcal{G}_1(\psi) = \alpha_2\mathcal{G}_2(\psi)$): this theory corresponds to the standard sGB theory.
  2. $\alpha_2 = 0$: this theory corresponds to a purely Teleparallel theory where the dynamics are governed by $F(\psi)T_G$.
  3. $\alpha_3\mathcal{G}_3(\psi) = -\alpha_2\mathcal{G}_2(\psi)$ (or equivalently $\alpha_1 = 0$): this theory also corresponds to a purely Teleparallel theory where the dynamics are governed by $F(\psi)B_G$.

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- The second and third cases ($\alpha_3 \neq 0$) are new in the literature and they can only exist when one considers Teleparallel gravity.

- The theory gives us the possibility to have such an unstable mode if

$$\frac{-4\alpha_3\ddot{\mathcal{G}}_3(\psi_0)+6\alpha_2\ddot{\mathcal{G}}_2(\psi_0)+5\beta M^2}{20\beta M^3}<0\,.$$

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- For our theory, spontaneous scalarization can occur in our theory for a much larger choice of parameters for different masses.
- Contrary to the sGB case, where scalarization of non-rotating black holes was possible only for $\alpha_2<0$, we can have scalarization for different signs of the coupling parameters.

- When $\alpha_3\dot{\mathcal{G}}_3 \neq \alpha_2\dot{\mathcal{G}}_2$, the scalar field at the horizon must satisfy [12]

$$\begin{aligned}\psi'_H \quad = \quad & \frac{r_H}{4(\alpha_2\dot{\mathcal{G}}_2 - \alpha_3\dot{\mathcal{G}}_3)}\left(1 \pm \frac{1}{\beta}\Big[\beta^2 + \frac{32(\alpha_3\dot{\mathcal{G}}_3 - \alpha_2\dot{\mathcal{G}}_2)}{r_H^8}\Big\{32\alpha_3^2\dot{\mathcal{G}}_3^2(\alpha_3\dot{\mathcal{G}}_3 - \alpha_2\dot{\mathcal{G}}_2)\right. \\ & \left. +\beta r_H^4(3\alpha_2\dot{\mathcal{G}}_2 + \alpha_3\dot{\mathcal{G}}_3)\Big\}\Big]^{1/2}\right) - \frac{8\alpha_3\dot{\mathcal{G}}_3}{\beta r_H^3}\,.\end{aligned}$$

[12] S. Bahamonde, D. D. Doneva, L. Ducobu, C. Pfeifer and S. S. Yazadjiev, Phys. Rev. D **107** (2023) no.10, 104013

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- This branch gives the correct condition for the Riemannian sGB case ($\alpha_3 = 0$) that has been used widely in the literature to solving the equations numerically:

$$\psi'_H = \frac{1}{4\alpha_2 r_H\dot{\mathcal{G}}_2}\left[r_H^2 \pm \sqrt{r_H^4 - \frac{96\alpha_2^2\dot{\mathcal{G}}_2^2}{\beta}}\right].$$

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- This analysis suggests that there are two different branches in Teleparallel sGB having asymptotically flat scalarized black hole configurations.

[12] S. Bahamonde, D. D. Doneva, L. Ducobu, C. Pfeifer and S. S. Yazadjiev, Phys. Rev. D **107** (2023) no.10, 104013

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This exponential function has one of the desired properties for scalarization, namely, it allows the GR solutions with a zero scalar field to be also solutions of the more general system of equations in Teleparallel gravity.

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- For this coupling, it was proven [13] that stable scalarized black hole solutions exist in the sGB case.

[13] D. D. Doneva and S. S. Yazadjiev, Phys. Rev. Lett. **120** (2018) no.13, 131103

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  In either of these two limiting cases, no contribution of the Riemannian Gauss-Bonnet term is present.
- These cases go beyond the classification of theories allowing for scalarization that is discussed in a recent Review[14]

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- We used the boundary condition derived before (at the horizon) and also we only concentrated on finding asymptotically flat solutions.
- First, In order to gain some intuition about the existence and behavior of black hole solutions let us start with discussing the bifurcation point, which corresponds to the point where Schwarzschild becomes unstable and new scalarized solutions originate, as well as the behavior of the scalarized black hole branches.

Numerical solutions - Mass and scalar charge case 1

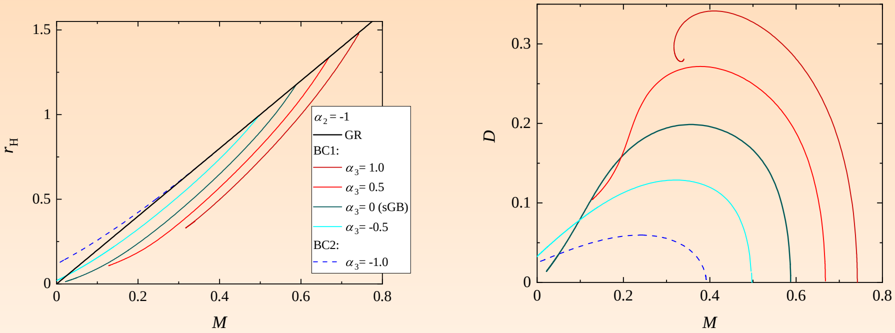

Figure: Setting $\alpha_2 = -1$ (the Riemannian sGB) and varying α_3

- With the increase of α_3 the point of bifurcation from the GR branch moves to large masses.

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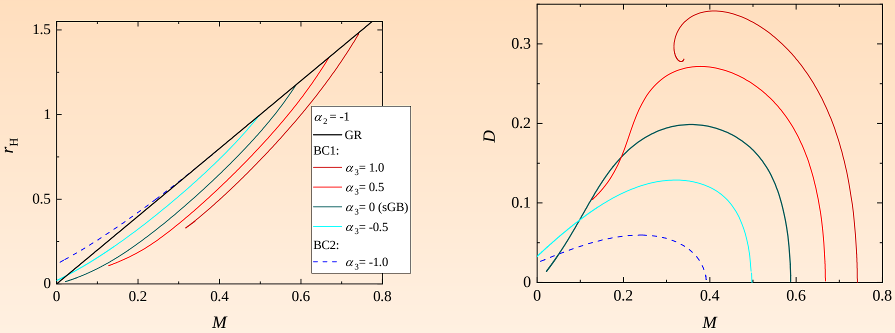

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- For larger α_3, the branch of scalarized solutions disappears at smaller masses.

Numerical solutions - Mass and scalar charge case 2

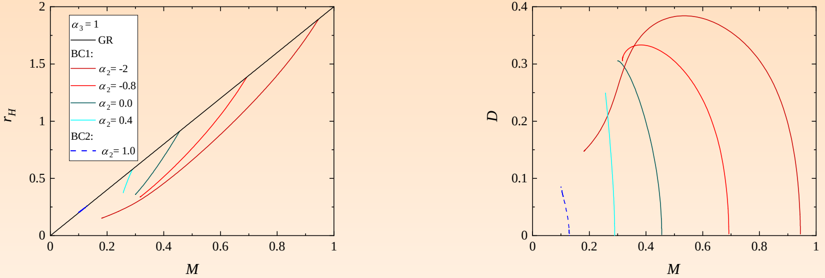

Figure: Setting $\alpha_3 = 1$ (the Teleparallel part) and varying α_2

- Contrary to the previous figure, larger α_2 move the bifurcation point to smaller masses

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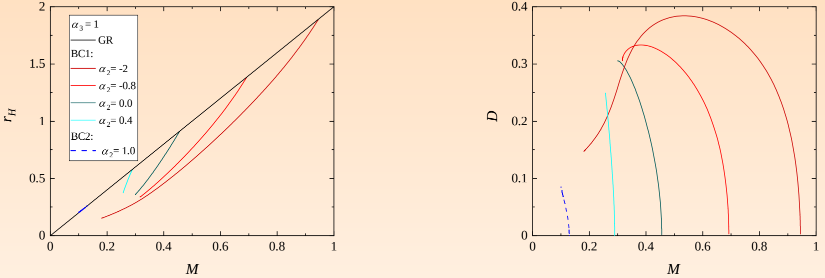

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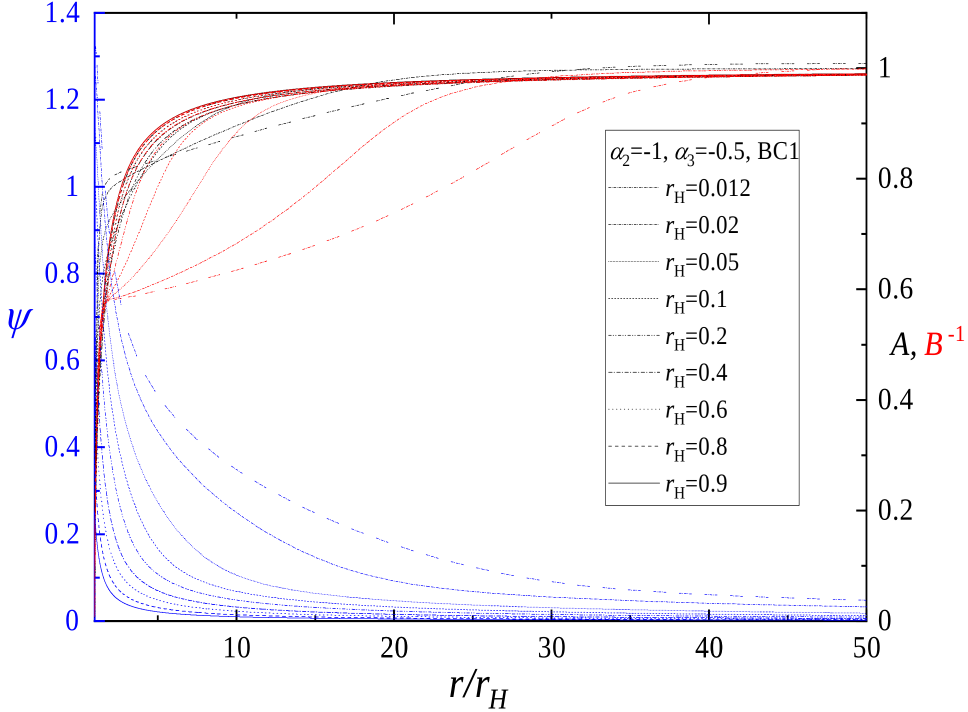
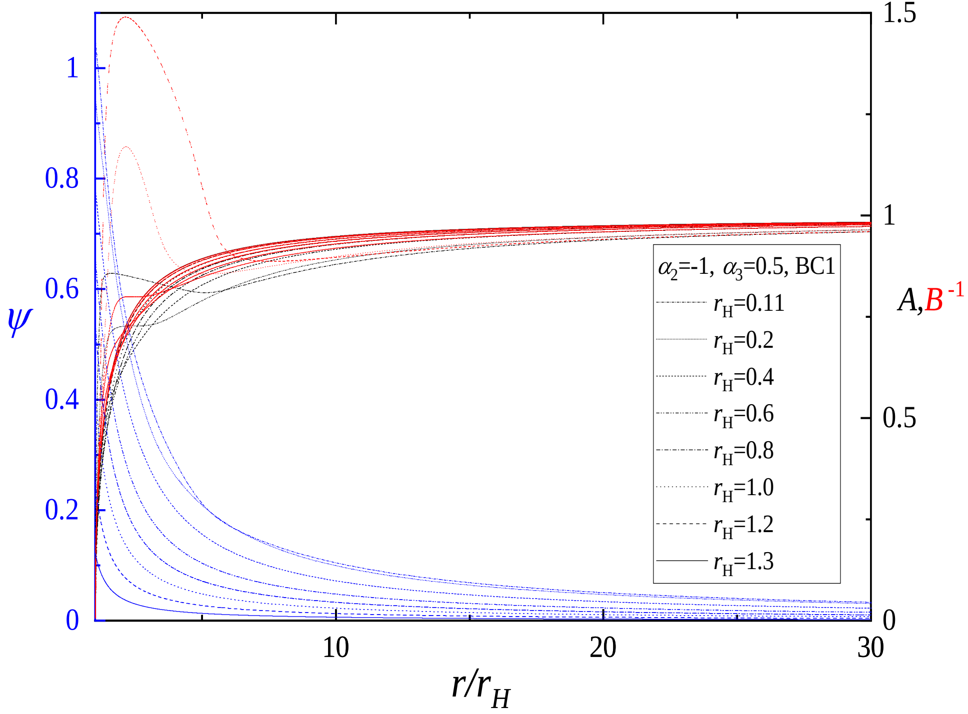
- Contrary to the previous figure, larger α_2 move the bifurcation point to smaller masses
- Even though this case offers a completely new type of scalarization, the behaviour of the solutions branches is qualitatively very similar to the sGB theory



α2=-1, α3=-0.5, BC1
rH=0.012
rH=0.02
rH=0.05
rH=0.1
rH=0.2
rH=0.4
rH=0.6
rH=0.8
rH=0.9
ψ
A, B -1
r/rH
1.4
1.2
1
0.8
0.6
0.4
0.2
0
10
20
30
40
50
0.8
0.6
0.4
0.2


ψ
A,B^{-1}
r/r_H
α_2=-1, α_3=0.5, BC1
r_H=0.11
r_H=0.2
r_H=0.4
r_H=0.6
r_H=0.8
r_H=1.0
r_H=1.2
r_H=1.3
0
0.2
0.4
0.6
0.8
1
10
20
30
0.5
1.5

New results with different couplings

- Last week, we just finished a new study within this theory[15] and we found that the real tetrad seems to be incompatible for constructing scalarized black holes.

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    - $\mathcal{G}_i = \psi^2$, which leads to black hole scalarization, i.e. Schwarzschild black hole is always a solution of the field equations but for small black hole masses, it becomes unstable giving rise to a spontaneously scalarized branch of solutions.
- Even though simpler compared to the exponential coupling considered before, the second choice leads to unstable black hole solutions in the Riemannian Gauss-Bonnet case. Interestingly, this observation might change for a strong enough torsional contribution.

[15] S. Bahamonde, D. D. Doneva, L. Ducobu, C. Pfeifer and S. S. Yazadjiev, [arXiv:2307.14720 [gr-qc]].

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  4. This has interesting implications: For example, if put in a binary, such a black hole will emit only very little scalar dipole radiation while the scalar field might influence the binary dynamics significantly.

Scalarized Black holes Gauss-Bonnet - Shift symmetric $\mathcal{G}_2 = \mathcal{G}_3 = \psi$

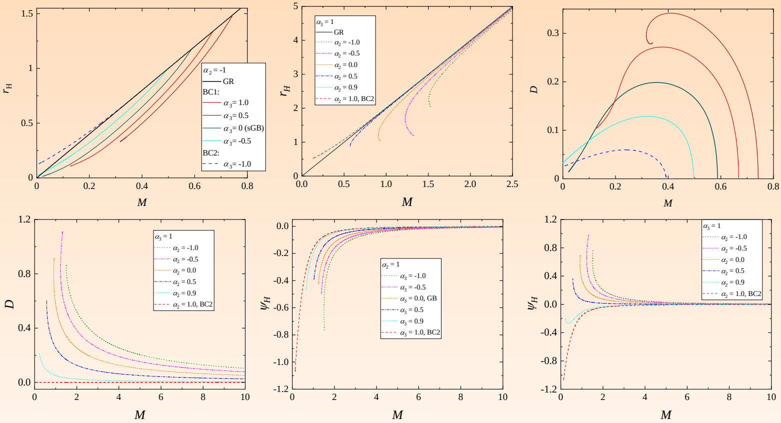

Figure: The radius of the horizon (*top panels)*, the scalar charge (*middle panels*), and the scalar field at the horizon (*bottom panels*). Schwarzschild is depicted with a solid black line in the top panels. The pure sGB case corresponds to $\alpha_2 = -1$ and $\alpha_3 = 0$. (S. Bahamonde, D. D. Doneva, L. Ducobu, C. Pfeifer and S. S. Yazadjiev, [arXiv:2307.14720 [gr-qc]].)

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  2. **Another indication of stability:** The horizon radius of the scalarized black holes, in this case, gets larger than the GR one contrary to all branches that turn right after bifurcation.

- Quadratic couplings is often avoided in numerical simulations since, in the Riemannian case at least, it leads to unstable black hole solutions.
- In addition, the branches of solutions are terminated shortly after the bifurcation point.
- In this case the GR-like black holes with zero scalar field are always solutions of the field equations and only in a certain range of black hole masses do additional scalarized solutions appear.
- Main results for this coupling:
  1. Interestingly, for certain combinations of α_2 and α_3, predominantly when the pure teleparallel term is stronger, the branch of solutions turn left after the bifurcation point instead of turning right. $\Longrightarrow$ indication of stable solutions!
  2. **Another indication of stability:** The horizon radius of the scalarized black holes, in this case, gets larger than the GR one contrary to all branches that turn right after bifurcation.
  3. our results indicate that the pure teleparallel term might potentially lead to a stabilization of the black holes for pure quadratic coupling.

Scalarized Black holes Gauss-Bonnet - Quadratic coupling $\mathcal{G}_2 = \mathcal{G}_3 = \psi^2$

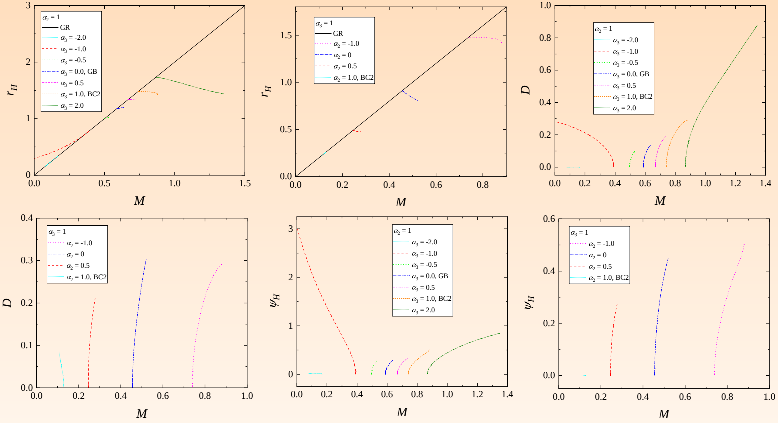

Figure: The radius of the horizon (*top panels*), the scalar charge (*middle panels*), and the scalar field at the horizon (*bottom panels*) as functions of mass for a quadratic coupling function.

Overview of the Talk

Gauss-Bonnet in Symmetric teleparallel Gravity

- Symmetric TG assumes that torsion and curvature are zero and nonmetricity $Q_{\alpha\mu\nu} = \nabla_\alpha g_{\mu\nu}$ is the responsible for gravity.

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- There is an equivalent description of gravity as GR but purely based on Q (similarly as TEGR).
- After a painful computation, we found the corresponding Gauss-Bonnet invariant in the a generic teleparallel theory (with torsion and nonmetricity):[16]

$$\begin{aligned}
\mathring{G} &= T_G + B_G \\
&= \delta^{\mu\,\nu\,\rho\,\sigma}_{\mu_1\mu_2\mu_3\mu_4} \Big[N^{\mu_1}{}_{\alpha\mu} N^{\alpha\mu_2}{}_{\nu} N^{\mu_3}{}_{\beta\rho} N^{\beta\mu_4}{}_{\sigma} - 2 N^{\mu_1\mu_2}{}_{\mu} N^{\mu_3}{}_{\alpha\nu} N^{\alpha}{}_{\beta\rho} N^{[\beta\mu_4]}{}_{\sigma} \\
&\qquad + 2 g_{\alpha\beta} N^{\mu_1\mu_2}{}_{\mu} N^{[\mu_3\alpha]}{}_{\nu} N^{\beta\mu_4}{}_{\gamma} N^{\gamma}{}_{\rho\sigma} + 2 g_{\alpha\beta} N^{\mu_1\mu_2}{}_{\mu} N^{[\mu_3\alpha]}{}_{\nu} N^{\beta\mu_4}{}_{\rho|\sigma} \\
&\qquad + 4 g_{\alpha\beta} g_{\gamma\delta} N^{\mu_1\mu_2}{}_{\mu} N^{[\mu_3\alpha]}{}_{\nu} N^{\mu_4\gamma}{}_{\rho} N^{(\delta\beta)}{}_{\sigma} \Big] . \\
&+ \frac{1}{\sqrt{-g}} \partial_\mu \left[\sqrt{-g}\, \delta^{\mu\,\nu\,\rho\,\sigma}_{\mu_1\mu_2\mu_3\mu_4} N^{\mu_1\mu_2}{}_{\nu} \left(N^{\mu_3}{}_{\lambda\rho} N^{\lambda\mu_4}{}_{\sigma} - \frac{1}{2} \mathring{R}^{\mu_3\mu_4}{}_{\rho\sigma} \right) \right] .
\end{aligned} \tag{20}$$

where

$$N^{\lambda}{}_{\mu\nu} = K^{\lambda}{}_{\mu\nu} + L^{\lambda}{}_{\mu\nu} = \frac{1}{2}\left(T^{\lambda}{}_{\mu\nu} - T_{\mu}{}^{\lambda}{}_{\nu} - T_{\nu}{}^{\lambda}{}_{\mu} \right) + \frac{1}{2}\left(Q^{\lambda}{}_{\mu\nu} - Q_{\mu}{}^{\lambda}{}_{\nu} - Q_{\nu}{}^{\lambda}{}_{\mu} \right) . \tag{21}$$

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Gauss-Bonnet in Symmetric teleparallel Gravity

- Then, we can focus on the Symmetric teleparallel gravity case (no torsion) and construct new Gauss-Bonnet theories:

$$\mathcal{S}_{\rm STsGB} = \frac{1}{2\kappa^2}\int \sqrt{-g}\Big[Q - \frac{1}{2}\beta\,\partial_\mu\psi\partial^\mu\psi + \alpha_1\mathcal{G}_1(\psi)T_G^{(Q)} + \alpha_2\mathcal{G}_2(\psi)B_G^{(Q)}\Big]\,\mathrm{d}^4x \tag{22}$$

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- Or, a generalization of the modified Gauss-Bonnet but in Symmetric tele:

$$\mathcal{S}_{f(Q,B_Q,T_G^{(Q)},B_G^{(Q)})} = \int \sqrt{-g}f(Q,B_Q,T_G^{(Q)},B_G^{(Q)})\mathrm{d}^4x\,. \qquad (23)$$

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- So far, we have found that there are much richer structure in spherical symmetry in the above theories.

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  3. What happens with spontenous scalarization for neutron stars?
  4. Within Symmetric TG, can one construct similar scalarization with the new Gauss-Bonnet that we derived?