Teleparallel gravity and its modifications

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Outline

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Tetrad fields

- Assuming that the manifold is differentiable: Define tetrads (or vierbein) $\{e_a\}$ (or $\{e^a\}$) which are the linear basis on the spacetime manifold.
- At each point of the spacetime, tetrads gives us basis for vectors on the tangent space.
- Notation: Greek letters $\rightarrow$ space-time indices;
  Latin letters $\rightarrow$ tangent space indices; ${E_a}^{\mu}$ is the inverse of the tetrad.
- Then for example $e_a = {E_a}^{\mu} dx_{\mu}$ and $e^a = {e^a}_{\mu} dx^{\mu}$.
- Tetrads satisfy the orthogonality condition: ${E_m}^{\mu} {e^n}_{\mu} = \delta^n_m$ and ${E_m}^{\nu} {e^m}_{\mu} = \delta^{\nu}_{\mu}$

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Tetrads and Special Relativity

- General basis $\{e_a\}$ satisfies $[e_a, e_b] = f^c{}_{ab} e_c$, where $f^c{}_{ab} = E_a{}^{\mu} E_b{}^{\nu} (\partial_\nu e^c{}_\mu - \partial_\nu e^c{}_\nu)$ is the coefficient of anholonomy (curl of the basis).
- Inertial frame (or holonomic frame) e'_a: $f'^a{}_{cd} = 0$ (global property)
- Special Relativity in Minkowski: Linear frames (trivial frames) gives us a relation between η_{ab} (tangent) and $\eta_{\mu\nu}$ (spacetime): $\eta_{ab} = \eta_{\mu\nu} E^\mu_a E^\nu_b$ and $\eta_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu$.
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Lorentz-rotated frame in SR

$$e^a{}_\mu = \Lambda^a{}_b e'^b_{\ \mu} = \partial_\mu x^a + w^a{}_{b\mu} x^b \equiv D_\mu x^a \,, \tag{1}$$

where $w^a{}_{b\mu} = \Lambda^a{}_e \partial_\mu \Lambda_b{}^e$ is known as the spin connection which represents inertial effects.

- Example free particles: inertial frame: $\frac{du'^a}{d\lambda} = 0$. This Eq. is not local Lorentz invariant
- However, in the anholonomic frame, changing $e^a{}_\mu = \Lambda^a{}_b e'^b_{\ \mu}$, one gets $\frac{du'^a}{d\lambda} + w^a{}_{b\mu} u^b u^\mu = 0$ which is Lorentz covariant!
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Gauge theory of translations - Teleparallel gravity

- TG is a gauge theory of translations: a field ψ transforms covariantly under a translation gauge $x^a \to x'^a + \xi^a(x^\mu)$ if a gauge potential ${B^a}_\mu = -\partial_\mu \xi^a$ is introduced.
- The translational gauge covariant derivative can be written as $e_\mu \psi = {e^a}_\mu \partial_a \psi$, and then

$$ {e^a}_\mu = \partial_\mu x^a + {B^a}_\mu \,, \tag{2} $$

where ${e^a}_\mu$ cannot be a trivial frame.
- This mean that one needs to replace a Minkowski metric to a Riemannian metric giving us

$$ g_{\mu\nu} = \eta_{ab} {e^a}_\mu {e^b}_\nu \,, \quad g^{\mu\nu} = \eta^{ab} {E^\mu}_a {E^\nu}_b \,. \tag{3} $$

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Gauge theory of translations - Inertial effects

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- By performing a Lorentz transformation $x^a \to \Lambda^a{}_b x^b$ one finds that the tetrad becomes

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Translation Field Strength

- Like any gauge theory, the field strength is the covariant rotational of the gauge potential

$$T^{a}{}_{\mu\nu} = D_{\mu}B^{a}{}_{\nu} - D_{\nu}B^{a}{}_{\mu}\,. \tag{5}$$

- Since $e^{a}{}_{\mu} = D_{\mu}x^{a} + B^{a}{}_{\mu}$, one can rewrite the field strength as

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- This expression coincides with the Torsion tensor!.
- It can be proved that the torsion tensor transforms covariantly under both diffeomorphisms and local Lorentz transformations and only represent gravitational effects.

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Connection in Teleparallel gravity

- Let us now introduce the so-called "Weitzenböck connection":

Weitzenböck connection

$$\tilde{\Gamma}^{\rho}{}_{\mu\nu} = E_a{}^{\rho} D_{\mu} e^a{}_{\nu} = E_a{}^{\rho}(\partial_{\mu} e^a{}_{\nu} + w^a{}_{b\mu} e^b{}_{\nu}) \,. \tag{7}$$

- By using this connection, one can express the torsion tensor as follows

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$$T^{\rho}{}_{\mu\nu} = \tilde{\Gamma}^{\rho}{}_{\nu\mu} - \tilde{\Gamma}^{\rho}{}_{\mu\nu} \,. \tag{8}$$

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- The Weitzenböck connection $\tilde{\Gamma}^{\rho}{}_{\nu\mu}$ is related to the Levi-Civita connection $\Gamma^{\rho}{}_{\nu\mu}$ via

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$$\tilde{\Gamma}^{\rho}{}_{\nu\mu} = \Gamma^{\rho}{}_{\nu\mu} + K^{\rho}{}_{\mu\nu}\,, \tag{9}$$

where $K^{\rho}{}_{\mu\nu} = \frac{1}{2}(T_{\mu}{}^{\rho}{}_{\nu} + T_{\nu}{}^{\rho}{}_{\mu} - T^{\rho}{}_{\mu\nu})$ is the contorsion tensor.

- In this connection, it is easy to verify that the spacetime is globally flat:

Curvature in Teleparallel gravity

$$R^{a}{}_{b\mu\nu}(\omega^{a}{}_{b\mu}) = \partial_{\mu}\omega^{a}{}_{b\nu} - \partial_{\nu}\omega^{a}{}_{b\mu} + \omega^{a}{}_{c\mu}\omega^{c}{}_{b\nu} - \omega^{a}{}_{c\nu}\omega^{c}{}_{b\mu} \equiv 0\,. \tag{10}$$

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Teleparallel action

- The teleparallel action is formulated based on a gravitational scalar called the torsion scalar T

$$S_{\rm TG} = \int \left[-T + 2\kappa^2 \mathcal{L}_{\rm m}\right] e\, d^4x\,. \qquad (11)$$

where $\kappa^2 = 8\pi G$, $e = \det(e_a^{\mu}) = \sqrt{-g}$, $\mathcal{L}_{\rm m}$ matter Lagrangian and $T = \frac{1}{4}T^{\rho}{}_{\mu\nu}T_{\rho}{}^{\mu\nu} + \frac{1}{2}T^{\rho}{}_{\mu\nu}T^{\nu\mu}{}_{\rho} - T^{\lambda}{}_{\lambda\mu}T_{\nu}{}^{\nu\mu}$.

- T and the scalar curvature R differs by a boundary term B as $R = -T + B$ so:

Equivalence between field equations

The teleparallel field equations are equivalent to the Einstein field equations.

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where $\kappa^2 = 8\pi G$, $e = \det(e^\mu_a) = \sqrt{-g}$, $\mathcal{L}_{\rm m}$ matter Lagrangian and $T = \frac{1}{4}T^{\rho}{}_{\mu\nu}T_{\rho}{}^{\mu\nu} + \frac{1}{2}T^{\rho}{}_{\mu\nu}T^{\nu\mu}{}_{\rho} - T^{\lambda}{}_{\lambda\mu}T_{\nu}{}^{\nu\mu}$.

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The teleparallel field equations are equivalent to the Einstein field equations.

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Teleparallel gravity vs General Relativity

Two completely equivalent ways of understanding gravity:

Connections and strength fields

G.R. $\Longrightarrow$ Levi-Civita connection $\Longrightarrow$ Curvature with vanishing torsion
TG $\Longrightarrow$ Weitzenböck connection $\Longrightarrow$ Torsion with vanishing curvature (flat).

How gravity is explained in both theories?

GR $\Longrightarrow$ Geometry (curvature of space-time) $\Longrightarrow$ geodesic equations
TG $\Longrightarrow$ Forces $\Longrightarrow$ Force equations as Maxwell eqs (no geodesic eq.).

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Gauge structure

GR $\Longrightarrow$ NO (only diffeomorphism)
TG $\Longrightarrow$ Gauge theory of the translations

Must have the equivalence principle?

GR $\Longrightarrow$ YES
TG $\Longrightarrow$ Can survive with or without

Can we separate inertia with gravity?

GR $\Longrightarrow$ NO (mixed) $\Longrightarrow$ No tensorial expression for the gravitational energy-momentum density
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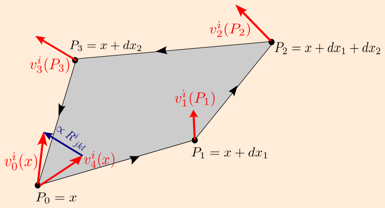
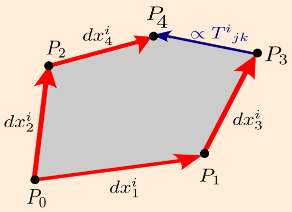
Geometrical differences

Curvature $\Longrightarrow$ how the tangent spaces roll along the curve.
Torsion $\Longrightarrow$ how tangent spaces twist about a curve when they are parallel transported





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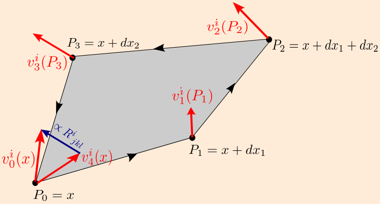

Figure: Transporting a vector in a closed trajectory creates a different vector

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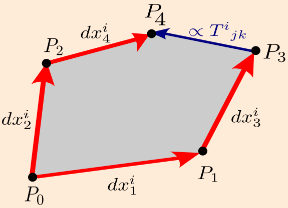

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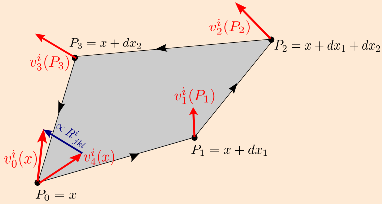

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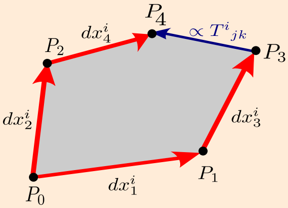

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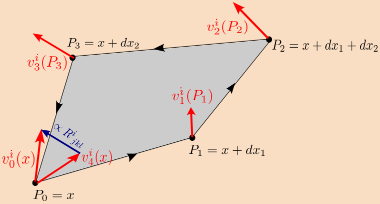

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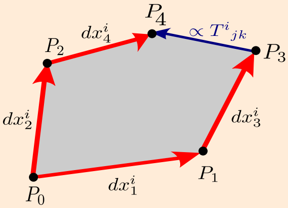

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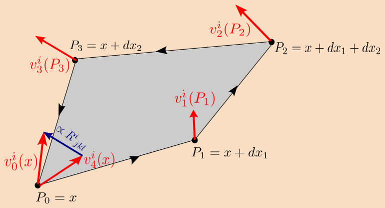

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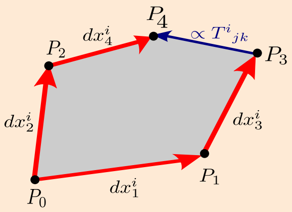

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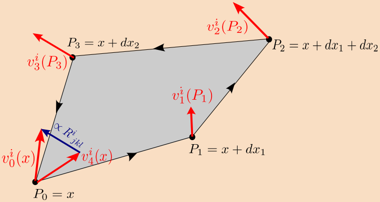

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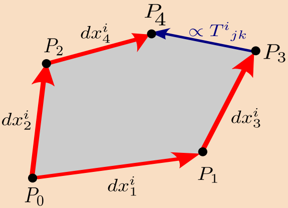

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Pure tetrad teleparallel

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$$T^{a}{}_{\mu\nu} = \partial_{\mu} e^{a}{}_{\nu} - \partial_{\nu} e^{a}{}_{\mu} = f^{a}{}_{\mu\nu} \qquad (12)$$

- Therefore, the torsion tensor and the scalar torsion T are not invariant under Local Lorentz transformations.
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Outline

1. Introduction to Teleparallel equivalent of general relativity
   - Basic concepts in teleparallel gravity
   - Gauge structure: translation group
   - Torsion, connection and action
   - Teleparallel gravity vs General Relativity
2. Modified teleparallel gravity theories
   - General properties
   - Modified teleparallel theories of gravity: $f(T,B)$ gravity
   - New classes of modified theories of gravity $f(T_i,B)$
   - Modified teleparallel theories of gravity: $f(T,B,T_G,B_G)$
3. Conclusions

Modified teleparallel theories of gravity

- As there are many modifications in GR, it is also possible to construct modifications to teleparallel gravity.
- If ones modifies pure tetrad teleparallel gravity, one finds theories which are not invariant under local LT.
- Two approaches for constructing modified teleparallel theories:

How to construct modified teleparallel theories?

- Both approaches give the same field equations but (1) is more theoretically correct. However, there is still a debate about how to find $w^a{}_{bc}$.
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Motivating from GR: $f(R)$ gravity

- A well studied modification of GR is $f(R)$ gravity, which has the following action (change $R \to f(R)$)

$f(R)$ gravity action

$$S_{f(R)} = \int (f(R) + 2\kappa^2 \mathcal{L}_{\rm m})\sqrt{-g}\, d^4x \,.$$

- Here, f is an arbitrary (sufficiently smooth) function of the Ricci scalar.
- Ricci scalar depends on second derivatives of the metric tensor $\to$ **Fourth order theory**.
- Very famous and explore theory which mimics cosmological observations.

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- The relationship between R and T are

Relationship between R and T

$$R = -T + \frac{2}{e}\partial_\mu(eT^\mu) = -T + B.$$

- Inspired by the later discussion, we define the action (Bahamonde et. al Phys.Rev. D92,10, 104042 (2015))

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$$S_{f(T,B)} = \int \left[f(T,B) + 2\kappa^2 L_{\rm m} \right] e\, d^4x \,,$$

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- It is possible to show that this theory has the correct limits:

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Lorentz invariance and conservation

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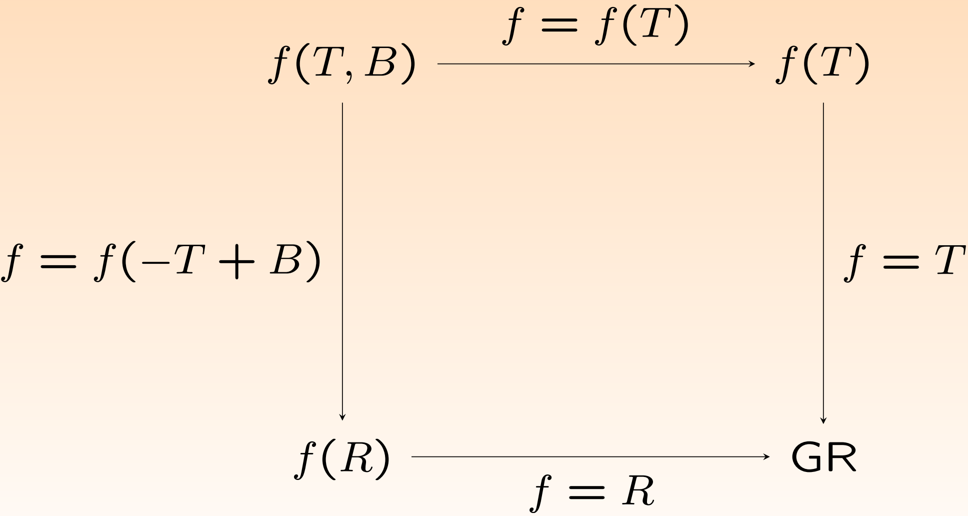
$f(T,B)$ diagram



Outline

1. Introduction to Teleparallel equivalent of general relativity
   - Basic concepts in teleparallel gravity
   - Gauge structure: translation group
   - Torsion, connection and action
   - Teleparallel gravity vs General Relativity
2. Modified teleparallel gravity theories
   - General properties
   - Modified teleparallel theories of gravity: $f(T,B)$ gravity
   - New classes of modified theories of gravity $f(T_i,B)$
   - Modified teleparallel theories of gravity: $f(T,B,T_G,B_G)$
3. Conclusions

Torsion decomposition

- The torsion tensor is

$$T^a{}_{\mu\nu}(e^a{}_\mu, \omega^a{}_{b\mu}) = \partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu + \omega^a{}_{b\mu} e^b{}_\nu - \omega^a{}_{b\nu} e^b{}_\mu \,, \quad (13)$$

which is generally non-vanishing, and transforms covariantly under both diffeomorphisms and local Lorentz transformations.

- It can be decomposed in three irreducible parts:

$$T_{\lambda\mu\nu} = \frac{2}{3}(t_{\lambda\mu\nu} - t_{\lambda\nu\mu}) + \frac{1}{3}(g_{\lambda\mu} v_\nu - g_{\lambda\nu} v_\mu) + \epsilon_{\lambda\mu\nu\rho} a^\rho \,, \quad (14)$$

where

$$v_\mu = T^\lambda{}_{\lambda\mu} \,, \quad a_\mu = \frac{1}{6}\epsilon_{\mu\nu\sigma\rho} T^{\nu\sigma\rho} \,,$$

$$t_{\lambda\mu\nu} = \frac{1}{2}(T_{\lambda\mu\nu} + T_{\mu\lambda\nu}) + \frac{1}{6}(g_{\nu\lambda} v_\mu + g_{\nu\mu} v_\lambda) - \frac{1}{3} g_{\lambda\mu} v_\nu \,.$$

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Teleparallel models - some famous modifications

- The first generalisation of TG was introduced in 1979, given by the following Lagrangian

New general Relativity Lagrangian (K. Hayashi and T. Shirafuji,1969)

$$\mathcal{L}_{\rm NGR} = e\Big(a_0 + a_1 T_{\rm ax} + a_2 T_{\rm ten} + a_3 T_{\rm vec}\Big) + 2\kappa^2 e\mathcal{L}_{\rm m}\,, \qquad (15)$$

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Lagrangian of the model

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Inclusion of the boundary term

- We can then include the boundary term and consider the Lagrangian

New class of modified TG with a boundary term

$$\mathcal{L} = e\, f(T_{\rm ax}, T_{\rm ten}, T_{\rm vec}, B) + 2\kappa^2 e\mathcal{L}_{\rm m}\,. \tag{18}$$

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Conformal transformations

- Let us rewrite the action by introducing two sets of four auxiliary fields $\phi_A = \{T_{\rm ax}, T_{\rm ten}, T_{\rm vec}, B\}$ and χ_A:

$$S = \frac{1}{2\kappa}\int \Big[f(\phi_A) + \chi_1(T_{\rm ax} - \phi_1) + \chi_2(T_{\rm ten} - \phi_2) + \chi_3(T_{\rm vec} - \phi_3) + \chi_4(B - \phi_4)\Big]\, e\, d^4x\,.$$

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- Next, let us apply a conformal transformation to the metric

$$\hat{g}_{\mu\nu} = \Omega^2(x) g_{\mu\nu}\,, \quad \hat{g}^{\mu\nu} = \Omega^{-2}(x) g^{\mu\nu}\,. \tag{19}$$

- The important quantities changes as follows

$$T_{\rm ax} = \Omega^2 \hat{T}_{\rm ax}\,, \tag{20}$$

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Minimal and non-minimal couplings

- To eliminate all the couplings between the scalar field and $\hat{T}^\mu$ (or equivalently B), the function must satisfy

$$f^{(0,0,1,1)}(\phi_A)^2 = f^{(0,0,0,2)} f^{(0,0,2,0)}(\phi_A)\,, \qquad (25)$$

$$f^{(0,1,1,0)}(\phi_A) f^{(0,0,0,2)}(\phi_A) = f^{(0,0,1,1)}(\phi_A) f^{(0,1,0,1)}(\phi_A)\,, \qquad (26)$$

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which gives us a non-minimally coupled theory between T_i and a scalar field.

Minimal and non-minimal couplings

- To obtain a theory where the scalar field is minimally coupled with the torsion scalar (an Einstein frame) we must ALSO impose

$$\Omega^2 = -\frac{2}{3}F_1(\phi_A) = -\frac{3}{2}F_2(\phi_A) = \frac{3}{2}F_3(\phi_A)\,. \qquad (28)$$

- The unique theory (besides TEGR) which satisfies this condition plus the conditions mentioned in the previous slide is

Unique theory with an Einstein frame

$$f(T_{\rm ax}, T_{\rm ten}, T_{\rm vec}, B) = \tilde{f}\Big(-\frac{3}{2}T_{\rm ax} - \frac{2}{3}T_{\rm ten} + \frac{2}{3}T_{\rm vec} + B\Big) = \tilde{f}(R)\,. \qquad (29)$$

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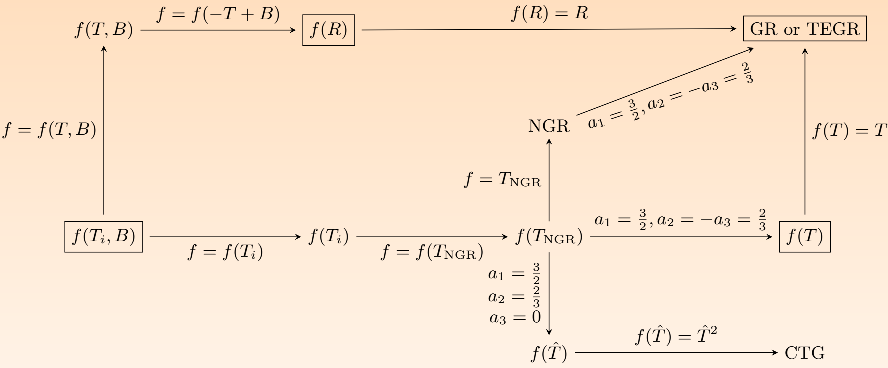

Figure: Relationship between different modified gravity models and General Relativity where $T_i = (T_{\rm ax}, T_{\rm ten}, T_{\rm vec})$, $T_{\rm NGR} = a_1 T_{\rm ax} + a_2 T_{\rm ten} + a_3 T_{\rm vec}$ and $\tilde{T} = \frac{3}{2} T_{\rm ax} + \frac{2}{3} T_{\rm ten}$. The abbreviations NGR, CTG and TEGR mean new general relativity, teleparallel conformal gravity and teleparallel equivalent of general relativity respectively.

Outline

Gauss-Bonnet extension

- The Gauss-Bonnet term is a quadratic combination of the Riemann tensor and its contractions given by

Gauss-Bonnet term

$$G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda}\,.$$

- This term can be expressed in a similar way as before which simply reads as

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- Inspired by the later discussion, we define the action (Bahamonde et. al, Eur.Phys.J. C76 (2016) no.10, 578)

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$$S_{f(T,B,T_G,B_G)} = \int \left[f(T,B,T_G,B_G) + 2\kappa^2 \mathcal{L}_{\rm m} \right] e\, d^4x \,.$$

- This action is very general. For instance, we can recover $f = f(T, T_G)$ gravity or $f(-T + B, -T_G + B_G) = f(R, G)$ gravity.
- In Bahamonde et. al, Eur.Phys.J. C76 (2016) no.10, 578 , an additional extension where the trace of the energy-momentum tensor $\mathcal{T} = E_a^\beta \mathcal{T}_\beta^a$ was introduced too; $f(T, B, T_G, B_G, \mathcal{T})$.

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Good tetrads: Example in FRWL Cosmology with $k=+1$

- Since T, B, T_G and B_G are not invariant under local Lorentz transformations $\Longrightarrow$ One needs to be careful with the choice of the tetrad.
- The simplest tetrad field which yields the FRWL metric $ds^2 = -dt^2 + a(t)^2\left[\frac{1}{1-kr^2}dr^2 + d\Omega^2\right]$ is a diagonal one given by

$$e^a_\mu = \text{diag}\left(1, a(t)/\sqrt{1-kr^2}, a(t)r, a(t)r\sin\theta\right).$$

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- The metric is invariant under the Lorentz rotated transformation $e'^a_\mu = \Lambda^a{}_b e^b_\mu$, where

$$\Lambda^a{}_b = \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{R}(\alpha, \beta, \gamma) \end{pmatrix} ,$$

  is the local Lorentz transformation matrix. Here, R is the 3-dimensional rotation in the tangent space parametrised by three Euler angles $\alpha = \theta - \frac{\pi}{2}$, $\beta = \phi$ and $\gamma = \gamma(r)$.
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- Using the rotated tetrad, the torsion scalar and the boundary term are

$$T = -\frac{4}{a^2}\Big(\frac{\sqrt{1-kr^2}}{r^2}\Big[r\gamma'\cos\gamma+\sin\gamma\Big]+\frac{1}{r^2}\Big)+6\frac{\dot{a}^2}{a^2}+2\frac{k}{a^2}\,,$$
$$B = -\frac{4}{a^2}\Big(\frac{\sqrt{1-kr^2}}{r^2}\Big[r\gamma'\cos\gamma+\sin\gamma\Big]+\frac{1}{r^2}\Big)+6\frac{\ddot{a}}{a}+12\frac{\dot{a}^2}{a^2}+8\frac{k}{a^2}\,.$$

- For the closed case $k = +1$, we need to set $\gamma(r) = -\operatorname{arcsinh}\left(\sqrt{1+r^2}\right)$ to have T and B position independent (also T_G and B_G) giving us in this frame:

$$T = 6H^2 + \frac{2}{a^2}\,, B = 6(H^2+2\dot{H}) + \frac{8}{a^2}\,, T_G = -\frac{24\ddot{a}}{a}(H^2 - \frac{1}{a^2})\,, B_G = \frac{48\ddot{a}}{a^3}\,, \tag{30}$$

  where $H = \dot{a}/a$ is the Hubble parameter.
- Clearly, $R = -T + B = 6(H^2 + 2\dot{H} + 1/a^2)$ and $G = -T + B_G = 24\ddot{a}(H^2 + 1/a^2)/a$ as expected!

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- Using the rotated tetrad in a closed universe with the correct γ, the field equation are given by

$$f+\frac{6\dot{a}\dot{f}_B}{a}-\frac{6f_B\left(a\ddot{a}+2\dot{a}^2\right)}{a^2}-\frac{12\dot{a}^2f_T}{a^2}-\frac{48f_{B_G}\ddot{a}}{a^3}+\frac{48\dot{a}\dot{f}_{B_G}}{a^3}$$
$$-\frac{24\left(\dot{a}^2-1\right)\dot{a}\dot{f}_{T_G}}{a^3}+\frac{24f_{T_G}\left(\dot{a}^2-1\right)\ddot{a}}{a^3}=2\kappa\,\rho\,,$$
$$f-\frac{48f_{B_G}\ddot{a}}{a^3}-\frac{4\dot{a}\dot{f}_T}{a}+\frac{8(1-\dot{a}^2)\ddot{f}_{T_G}}{a^2}-\frac{6f_B\left(a\ddot{a}+2\dot{a}^2\right)}{a^2}$$
$$-\frac{f_T\left(4a\ddot{a}+8\dot{a}^2-4\right)}{a^2}-\frac{16\dot{a}\ddot{a}\dot{f}_{T_G}}{a^2}+\frac{24f_{T_G}\left(\dot{a}^2-1\right)\ddot{a}}{a^3}$$
$$+\frac{16\ddot{f}_{B_G}}{a^2}+2\ddot{f}_B=-2\kappa\,p\,.$$

- If $f(T,B,T_G,B_G)=\mathfrak{f}(-T+B,-T_G+B_G)=\mathfrak{f}(R,G)$ we recover the usual Gauss-Bonnet equations $\mathfrak{f}(R,G)$.

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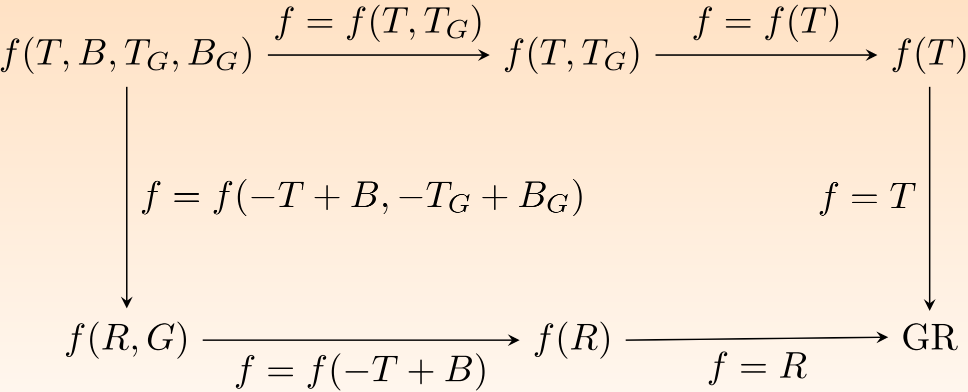
$f(T, B, T_G, B_G)$ diagram



Conclusions

- Teleparallel gravity is a gauge theory of the translation group which leads a special connection with zero cuvature and non-zero torsion (Weitzbröck connection).
- TG is equivalent on its field equations to GR, but its physical interpretation is different.
- Two alternative ways of understanding gravity: either GR (torsion zero and curvature non-zero) or TG (curvature zero and torsion non-zero).
- Modified teleparallel theories of gravity have been very popular nowadays. We have explored many of them and how are they connected (also with standard theories!): $f(T), f(T,B), f(T_i,B), f(T,B,T_G,B_G)$, among others.

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- Modified teleparallel theories (in the pure tetrad formalism) are not invariant under Lorentz transformations but one can solve this problem:
  1. Starting with a non-pure tetrad formalism with spin connection being different than zero.
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