

Black holes solutions in metric-affine gravity with dynamical torsion and nonmetricity

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Outline

- 1 Brief introduction to Metric-affine gravity
 - Basic quantities
 - Dynamics
- 2 MAG models with dynamical torsion and nonmetricity
 - Spherical symmetry
 - Observational constraints
 - Axial symmetry

Fundamental variables and characteristic tensors

- In the most general metric-affine setting, the fundamental variables are a **metric** $g_{\mu\nu}$ (10 comp.) as well as the coefficients $\tilde{\Gamma}^{\rho}_{\mu\nu}$ (64 comp.) of an **affine connection**.

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Curvature	$\tilde{R}^\mu{}_{\nu\rho\sigma} = \partial_\rho \tilde{\Gamma}^\mu{}_{\nu\sigma} - \partial_\sigma \tilde{\Gamma}^\mu{}_{\nu\rho} + \tilde{\Gamma}^\mu{}_{\tau\rho} \tilde{\Gamma}^\tau{}_{\nu\sigma} - \tilde{\Gamma}^\mu{}_{\tau\sigma} \tilde{\Gamma}^\tau{}_{\nu\rho}$
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Torsion	$\tilde{T}^\mu{}_{\nu\rho} = \tilde{\Gamma}^\mu{}_{\rho\nu} - \tilde{\Gamma}^\mu{}_{\nu\rho}$
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Nonmetricity	$\tilde{Q}_{\mu\nu\rho} = \tilde{\nabla}_\mu g_{\nu\rho} = \partial_\mu g_{\nu\rho} - \tilde{\Gamma}^\sigma{}_{\nu\mu} g_{\sigma\rho} - \tilde{\Gamma}^\sigma{}_{\rho\mu} g_{\nu\sigma}$
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Dynamics of metric-affine geometry

- Gravitational action with dynamical torsion and nonmetricity:

$$S = \int d^4x \sqrt{-g} \left[\mathcal{L}_m - \frac{1}{16\pi} \mathcal{L}_g(\tilde{\mathcal{R}}, \mathcal{T}, \mathcal{Q}) \right]. \quad (2)$$

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- Correspondence between geometry and matter:

$$\frac{\delta S_g}{\delta e^a{}_\nu} = 16\pi \theta_a{}^\nu, \quad (3)$$

$$\frac{\delta S_g}{\delta \omega^a{}_{b\nu}} = 16\pi \Delta_a{}^{b\nu}. \quad (4)$$

Here $\theta_a{}^\nu$ is the energy-momentum tensor (canonical) and $\Delta_a{}^{b\nu}$ is the hypermomentum density tensor.

MAG models with dynamical torsion and nonmetricity

- Quadratic gravitational action with dynamical torsion and nonmetricity in Weyl-Cartan geometry ($Q_{\lambda\mu\nu} = g_{\mu\nu}W_\lambda$)

$$\begin{aligned}
 S = \int d^4x \sqrt{-g} \left\{ \mathcal{L}_m + \frac{1}{64\pi} \left[-4R - 6d_1 \tilde{R}_{\lambda[\rho\mu\nu]} \tilde{R}^{\lambda[\rho\mu\nu]} \right. \right. \\
 - 9d_1 \tilde{R}_{\lambda[\rho\mu\nu]} \tilde{R}^{\mu[\lambda\nu\rho]} + 8d_1 \tilde{R}_{[\mu\nu]} \tilde{R}^{[\mu\nu]} + \frac{1}{8} (32e_1 + 8e_2 + 17d_1) \tilde{R}^\lambda{}_{\lambda\mu\nu} \tilde{R}^\rho{}_{\rho}{}^{\mu\nu} \\
 \left. \left. - 7d_1 \tilde{R}_{[\mu\nu]} \tilde{R}^\lambda{}_{\lambda}{}^{\mu\nu} + 3(1 - 2a_2) T_{[\lambda\mu\nu]} T^{[\lambda\mu\nu]} \right] \right\}.
 \end{aligned}$$

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- Absence of a general Birkhoff's theorem in MAG: new spherically and axially symmetric vacuum solutions with independent dynamical torsion and nonmetricity fields^{1,2}.

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Spherical symmetry

- Metric, torsion and nonmetricity in spherically symmetric space-times (#2 + #8 + #2 = #12):

$$\mathcal{L}_\xi g_{\mu\nu} = \mathcal{L}_\xi T^\lambda{}_{\mu\nu} = \mathcal{L}_\xi W_\mu = 0 \implies \mathcal{L}_\xi \tilde{R}_{\lambda\rho\mu\nu} = 0$$

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- By solving these equations we find that torsion and nonmetricity behave as

$$\begin{aligned} T^t{}_{tr} &= a(r), & T^r{}_{tr} &= b(r), & T^{\theta_k}{}_{t\theta_k} &= f(r), & T^{\theta_k}{}_{r\theta_k} &= g(r) \\ T^{\theta_k}{}_{t\theta_l} &= e^{a\theta_k} e^b{}_{\theta_l} \epsilon_{ab} d(r), & T^{\theta_k}{}_{r\theta_l} &= e^{a\theta_k} e^b{}_{\theta_l} \epsilon_{ab} h(r), \\ T^t{}_{\theta_k\theta_l} &= \epsilon_{kl} k(r) \sin \theta_1, & T^r{}_{\theta_k\theta_l} &= \epsilon_{kl} l(r) \sin \theta_1, \\ W_\lambda &= (w_1(r), w_2(r), 0, 0), \end{aligned}$$

whereas the metric is in the standard spherically symmetric form:

$$ds^2 = \Psi_1(r) dt^2 - \frac{dr^2}{\Psi_2(r)} - r^2 (d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2).$$

Here, ϵ_{kl} is the Levi-Civita symbol in two dimensions.

Spherical symmetry - Solving the field equations

The field eqs are very involved. To solve them we use the following strategy:

- 1 **Imposing regularity:** In general, the solutions can have a singular behaviour. To ensure regularity, one can analyse the torsion/nonmetricity tensors referred to the rotated basis $\vartheta^a = \Lambda^a_b e^b$.

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One can write the gauge curvature $\mathcal{F}^a_{bc} = \vartheta^a_\lambda \vartheta_b^\mu \vartheta_c^\nu T^\lambda_{\nu\mu}$ related to the torsion/nonmetricity tensor in this orthogonal coframe.

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Regularity restricts the initial arbitrariness of the torsion components and the Weyl vector by imposing the relations

$$\begin{aligned} b(r) &= a(r) \sqrt{\Psi_1(r)\Psi_2(r)}, & f(r) &= -g(r) \sqrt{\Psi_1(r)\Psi_2(r)}, \\ d(r) &= -h(r) \sqrt{\Psi_1(r)\Psi_2(r)}, & l(r) &= k(r) \sqrt{\Psi_1(r)\Psi_2(r)}, \\ w_1(r) &= -w_2(r) \sqrt{\Psi_1(r)\Psi_2(r)}. \end{aligned}$$

Spherical symmetry - Solving the field equations

- 2 **Solve the weak field limit:** The weak field limit of the field equations become

$$\begin{aligned}\nabla_{\rho}\nabla_{\lambda}T^{\lambda\rho}{}_{\mu} + \nabla_{\rho}\nabla^{\rho}T^{\lambda}{}_{\mu\lambda} - \nabla_{\rho}\nabla_{\mu}T^{\lambda\rho}{}_{\lambda} &= 0, \\ \nabla_{\mu}\tilde{R}^{\lambda}{}_{\lambda}{}^{\mu\nu} &= 0.\end{aligned}$$

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These equations can be solved, yielding

$$\begin{aligned}w_1(r) &= -\kappa_d \int \sqrt{\frac{\Psi_1(r)}{\Psi_2(r)}} \frac{dr}{r^2}, \\ b(r) &= rf'(r) + f(r) + \frac{\kappa_d}{2r} \sqrt{\frac{\Psi_1(r)}{\Psi_2(r)}},\end{aligned}$$

where κ_d is an integration constant which represents the dilaton charge.

Spherical symmetry - Solving the field equations

- 1 The final solution for the metric behaves as
Reissner-Nordström

$$g_{tt} = -1/g_{rr} \equiv \Psi(r) = 1 - \frac{2m}{r} + \frac{d_1 \kappa_s^2 - 4e_1 \kappa_{d,e}^2}{r^2}. \quad (5)$$

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$$W_\mu = \frac{\kappa_{d,e}}{r} (1, -1/\Psi(r), 0, 0). \quad (6)$$

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- 3 Torsion sector:

$$\bar{S}^a = -\frac{\kappa_s}{r} (1, 1, 0, 0), \quad (7)$$

$$\bar{T}_2^{abc} = \frac{\kappa_s}{3r} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}. \quad (8)$$

Dilation and spin charges

What do κ_s (dilation charge) and $\kappa_{d,e}$ (spin charge) physically represent?

Point 1 - Hypermomentum density

In the geometric scheme of MAG, not only an energy-momentum tensor of matter arises as source of curvature, but also a hypermomentum density tensor which operates as source of torsion and nonmetricity.

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Point 2 - Dilation and spin charges

In Weyl-Cartan geometry, hypermomentum density tensor splits into spin and dilation currents, which carry their own charges and provide a RN solution.

Dilation and spin charges

When these charges might be important?

Significant effects are contemplated only around **extreme gravitational systems**, such as **neutron stars** with intense magnetic fields and sufficiently oriented elementary spins or **black holes** endowed with spin and dilation charges.

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Significant effects are contemplated only around **extreme gravitational systems**, such as **neutron stars** with intense magnetic fields and sufficiently oriented elementary spins or **black holes** endowed with spin and dilation charges.

Quantum nature

The **intrinsic hypermomentum of matter** is purely quantum since it vanishes in the rest of ordinary matter sources (e.g. Dirac fermions).

Observational constraints

- Let us now consider the case where the effect of torsion dominates over the contribution of nonmetricity.

³S. Bahamonde and J. Gigante Valcarcel, Eur. Phys. J. C **81** (2021) no.6, 495.

Observational constraints

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- Indeed, due to the presence of a magnetic field in white dwarfs, it is expected that Sirius B can have sufficiently oriented elementary spins in comparison with an effective dilation charge, therefore, $\kappa_{s,\text{SiriusB}} \gg \kappa_{d,\text{SiriusB}}$.

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- **Perihelion shift+ Gravitational redshift:** Assuming the same approximation in Sgr A* and considering the universality of the coupling constant d_1 , we find³

$$1.396 \cdot 10^{10} \leq \frac{\kappa_{s,\text{SgrA*}}}{\kappa_{s,\text{SiriusB}}} \leq 1.688 \cdot 10^{10} .$$

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- To the best of our knowledge, this bound provides the first observational comparison between the spin charges of a supermassive black hole and a degenerate star.

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Extension to axisymmetric space-times

- Metric, torsion and nonmetricity tensors in symmetric space-times:

$$\mathcal{L}_\xi g_{\mu\nu} = \mathcal{L}_\xi T^\lambda{}_{\mu\nu} = \mathcal{L}_\xi Q^\lambda{}_{\mu\nu} = 0 \implies \mathcal{L}_\xi \tilde{R}^\lambda{}_{\rho\mu\nu} = 0. \quad (9)$$

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- Stationary and axisymmetric space-times⁴:

$$\#10 \rightarrow \#4 \left\{ \begin{array}{l} ds^2 = \Psi_1(r, \vartheta) dt^2 - \frac{dr^2}{\Psi_2(r, \vartheta)} \\ - r^2 \Psi_3(r, \vartheta) \left[d\vartheta^2 + \sin^2 \vartheta (d\varphi - \Psi_4(r, \vartheta) dt)^2 \right] \end{array} \right. ;$$

$$\#24 \left\{ T^\lambda{}_{\mu\nu} = T^\lambda{}_{\mu\nu}(r, \vartheta) \right. \quad (10)$$

$$\#4 \left\{ W_\mu = (W_t(r, \vartheta), W_r(r, \vartheta), W_\vartheta(r, \vartheta), W_\varphi(r, \vartheta)) \right. \quad (11)$$

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Extension to axisymmetric space-times

- Rotating Kerr-Newman metric structure:

$$\begin{aligned} ds^2 = & \Psi(r, \vartheta) dt^2 - \frac{r^2 + a^2 \cos^2 \vartheta}{(r^2 + a^2 \cos^2 \vartheta) \Psi(r, \vartheta) + a^2 \sin^2 \vartheta} dr^2 \\ & - (r^2 + a^2 \cos^2 \vartheta) d\vartheta^2 + 2a (1 - \Psi(r, \vartheta)) \sin^2 \vartheta dt d\varphi \\ & - \sin^2 \vartheta [r^2 + a^2 + a^2 (1 - \Psi(r, \vartheta)) \sin^2 \vartheta] d\varphi^2, \end{aligned} \quad (12)$$

$$\Psi(r, \vartheta) = 1 - \frac{[2mr + 4e_1(\kappa_{d,e}^2 + \kappa_{d,m}^2) - d_1 \kappa_s^2]}{r^2 + a^2 \cos^2 \vartheta}. \quad (13)$$

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 & - (r^2 + a^2 \cos^2 \vartheta) d\vartheta^2 + 2a (1 - \Psi(r, \vartheta)) \sin^2 \vartheta dt d\varphi \\
 & - \sin^2 \vartheta [r^2 + a^2 + a^2 (1 - \Psi(r, \vartheta)) \sin^2 \vartheta] d\varphi^2, \quad (12)
 \end{aligned}$$

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- Field strength tensors:

$$\begin{aligned}
 \bar{R}_{[\mu\nu]} = & \frac{1}{12} \varepsilon^\lambda{}_{\sigma\mu\nu} \nabla_\lambda \bar{S}^\sigma + \frac{1}{2} \nabla_\lambda \bar{t}^\lambda{}_{\mu\nu}; \quad \tilde{R}^\lambda{}_{\lambda\mu\nu} = 4 \nabla_{[\nu} W_{\mu]}; \\
 \bar{R}^\lambda{}_{[\mu\nu\rho]} = & \frac{1}{6} \varepsilon^\lambda{}_{\sigma[\rho\nu} \nabla_{\mu]} \bar{S}^\sigma + \nabla_{[\mu} \bar{t}^\lambda{}_{\rho\nu]} + \frac{1}{4} \varepsilon^\lambda{}_{\omega\sigma[\rho} \bar{t}_1^\sigma{}_{\mu\nu]} \bar{S}^\omega \\
 & - \frac{1}{18} \varepsilon_{\sigma\mu\nu\rho} \bar{T}_1^\lambda{}_{\bar{S}}^\sigma. \quad (14)
 \end{aligned}$$

Extension to axisymmetric space-times

- Nonmetricity sector:

$$\begin{aligned}w_1(r, \vartheta) &= \frac{\kappa_{d,e} r - a \kappa_{d,m} \cos \vartheta}{r^2 + a^2 \cos^2 \vartheta}, & w_3(r, \vartheta) &= 0, \\w_2(r, \vartheta) &= - \frac{\kappa_{d,e} r}{(r^2 + a^2 \cos^2 \vartheta) \Psi(r, \vartheta) + a^2 \sin^2 \vartheta}, \\w_4(r, \vartheta) &= \kappa_{d,m} \left(\frac{r^2 + a^2}{r^2 + a^2 \cos^2 \vartheta} \cos \vartheta - \gamma \right) - a \frac{\kappa_{d,e} r \sin^2 \vartheta}{r^2 + a^2 \cos^2 \vartheta}.\end{aligned}\quad (15)$$

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 \end{aligned}$$

- Torsion sector (decoupling limit between the spin and the orbital angular momentum $|a\kappa_s| \ll 1$):

$$\bar{S}^a = -\frac{\kappa_s}{r}(1, 1, 0, 0) + \mathcal{O}(a\kappa_s), \quad (16)$$

$$\bar{T}_2^{abc} = \frac{\kappa_s}{3r} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} + \mathcal{O}(a\kappa_s). \quad (17)$$

Gravitational spin-orbit interaction

- We found a solution in the decoupling limit $a\kappa_s \ll 1$, which ensures that the Maxwell equation and closure conditions are fulfilled by the field strength tensors of torsion

$$\nabla_\lambda \tilde{R}^\lambda{}_{[\rho\mu\nu]} = \nabla_\mu \tilde{R}^{[\mu\nu]} = 0, \quad \nabla_{[\sigma} \tilde{R}_{\lambda[\rho\mu\nu]} = \nabla_{[\lambda} \tilde{R}_{\mu\nu]} = 0.$$

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Possible new effects in the decoupling limit

The dynamics of torsion and nonmetricity alters the geometry of the space-time \implies

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- **Gravitational spin-orbit interaction:**

$$\mathcal{H}_{\text{LS}} = \frac{1}{m_e^2 r} \frac{\partial V}{\partial r} \mathbf{L} \cdot \mathbf{S} \approx \frac{d_1}{2r} \frac{\partial g_{tt}}{\partial r} a\kappa_s \cos \vartheta \quad (18)$$

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- In progress: we are constructing Plebanski-Demianski uniformly accelerated rotating black hole solutions with NUT parameter, electromagnetic charges and a cosmological constant.
- Future: search of a gravitational spin-orbit interaction in MAG beyond the Kerr-Newman space-time (MAG is the main candidate to describe a spin-orbit interaction beyond GR).