Black hole solutions in Teleparallel gravity

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Outline

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Fundamental variables and characteristic tensors

- In the most general metric-affine setting, the fundamental variables are a **metric** $g_{\mu\nu}$ (10 comp.) as well as the coefficients $\hat{\Gamma}^{\rho}{}_{\mu\nu}$ (64 comp.) of an **affine connection.**

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Torsion	$\hat{T}^{\mu}{}_{\nu\rho} = \hat{\Gamma}^{\mu}{}_{\nu\rho} - \hat{\Gamma}^{\mu}{}_{\rho\nu}$
Nonmetricity	$\hat{Q}_{\mu\nu\rho} = \hat{\nabla}_{\mu}g_{\nu\rho} = \partial_{\mu}g_{\nu\rho} - \hat{\Gamma}^{\sigma}{}_{\nu\mu}g_{\sigma\rho} - \hat{\Gamma}^{\sigma}{}_{\rho\mu}g_{\nu\sigma}$

Tetrads and spin connection

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where η_{ab} is the Minkowski metric.

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Weitzenböck connection: Curvature is zero and $\nabla_\mu g_{\nu\rho} = 0$:

Teleparallel connection - Weitzenböck

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In TG, it is always possible to find a frame such that $\omega^a{}_{b\mu} = 0$, but this is a gauge choice, so only some tetrads are compatible with this.

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- The torsion tensor is generally non-vanishing, and transforms covariantly under both diffeomorphisms and local Lorentz transformations.

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- One has that
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- **Be careful here!** The general curvature $R^{\lambda}{}_{\mu\sigma\nu} \equiv 0$, not $\mathring{R}^{\lambda}{}_{\mu\sigma\nu} \neq 0$

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- The Ricci scalar computed from the Levi-Civita connection $\overset{\circ}{R}$ differs from the scalar torsion T by a boundary term B.

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where $\kappa^2 = 8\pi G$ and $L_{\rm m}$ is any matter Lagrangian.

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Equivalence between field equations

The field equations arising from S_{TEGR} are equivalent to the Einstein field equations.

Two different ways of understanding gravity

Coupling to matter

In TG, no direct matter coupling to the teleparallel connection is introduced, in order to preserve the weak equivalence principle $\Longrightarrow$ matter fields retain their universal **coupling to the metric** and possibly its **Levi-Civita connection** (in the case of spinor fields).

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TEGR has the same equations as GR, so **CLASSICALLY** it is impossible to make any observation to distinguish between them. All **classical experiments** already done, that have confirmed GR, also can be understood as a confirmation of TEGR.

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See our review S. Bahamonde et al.,"Teleparallel Gravity: From Theory to Cosmology," [arXiv:2106.13793 [gr-qc]] for more details.

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Modified teleparallel theories

What happens if we modify TEGR?

If we modify the TEGR action, a priori there is no equivalence between modified theories from GR and modified Teleparallel theories.

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Are Teleparallel theories Lorentz invariant?

There is a lot of misconceptions in the literature since TEGR and their first modifications were proposed in the pure-tetrad formalism, which assumes that $\omega^{a}{}_{b\mu} = 0$ globally. By assuming this choice, TEGR is pseudo local Lorentz invariant (invariant up to a boundary term) and in modified TG, there is a breaking of the local Lorentz invariant.

Modified teleparallel theories

Is it possible to formulate Teleparallel theories in a fully invariant form?

Yes, and the way to do this is by incorporating the spin connection ${\omega^a}_{b\mu}$ in the formulation. If one introduces this quantity, the torsion tensor is always fully covariant (covariant under both diffeo and local Lorentz) and then, any action constructed from it will be fully invariant.

See[1]

[1] M. Kršák and E. N. Saridakis, Class. Quant. Grav. **33** (2016) no.11, 115009; M. Krssak, R. J. van den Hoogen, J. G. Pereira, C. G. Böhmer and A. A. Coley, Class. Quant. Grav. **36** (2019) no.18, 183001.

Important properties of Teleparallel theories

- Teleparallel theories have the tetrads and spin connection as the fundamental variables, so that, one most commonly assumes an action which is of the form

$$\mathcal{S} = \mathcal{S}_{\mathsf{g}}[e, \omega] + \mathcal{S}_{\mathsf{m}}[e, \chi]\,,$$

where the gravitational part $\mathcal{S}_{\mathsf{g}}$ of the action depends on the tetrad $e^A{}_\mu$ and the spin connection $\omega^A{}_{B\mu}$, while the matter part depends on the tetrad $e^A{}_\mu$ and arbitrary matter fields χ^I, but not on the spin connection.

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- Particles (bosonic or femionic) follow the standard geodesic equation.

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- Variations of any action should be taken with respect to both variables (with the emphasis that the spin connection is non-arbitrary but always flat).

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- Variations of any action should be taken with respect to both variables (with the emphasis that the spin connection is non-arbitrary but always flat).
- Since $\omega^{A}{}_{B\mu}$ is a pure-gauge quantity, it can be shown that the the antisymmetric part of the field equations arising from variations w/r to the tetrads $e^{A}{}_{\mu}$ coincides with the variations of the action w/r to $\omega^{A}{}_{B\mu}$.

Important properties of Teleparallel theories

- Field equations are fully covariant (Lorentz and diffeo), and after finding them, it is possible to choose a gauge, known as the Weitzenböck gauge which is a special frame where ${\omega^A}_{B\mu} = 0$.

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- This gauge choice can be only taken after deriving the field equations and if one does this, only some tetrads will be compatible with this choice.

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Theories with scalar torsion and boundary term - $f(T,B)$

- One interesting modified TG theory is when one considers[2]

[2] S. Bahamonde, C. G. Böhmer and M. Wright, Phys. Rev. D **92** (2015) no.10, 104042

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$f(T,B)$ gravity action

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Theories with scalar torsion and boundary term - $f(T,B)$

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- There has been quite a lot of study about this theory in the last years.

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Field equations and spherical symmetry

- The separate tetrad and spin connection variations produce the field equations

$$W_{(\mu\nu)} = \kappa^2 \Theta_{\mu\nu}\,,$$

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- In GR the relation among them corresponds to the Bianchi identity $\overset{\circ}{R}{}^{\alpha}{}_{\beta[\lambda\mu;\nu]} = 0$, which leads to covariant conservation of the Einstein tensor.
- We showed that for $f(T,B)$, if the antisymmetric part of equations is satisfied, then the covariant divergence of equations of motion vanishes identically.

Field equations and spherical symmetry

Killing eqs:

$$\mathcal{L}_{Z_\zeta} e^A{}_\mu = -\lambda_\zeta^A{}_B e^B{}_\mu \,, \quad \mathcal{L}_{Z_\zeta} \omega^A{}_{B\mu} = \partial_\mu \lambda_\zeta^A{}_B + \omega^A{}_{C\mu} \lambda_\zeta^C{}_B - \omega^C{}_{B\mu} \lambda_\zeta^A{}_C \,.$$

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By solving these eqs in the Weitzenböck gauge we find

$$e^A{}_\nu = \begin{pmatrix} C_1 & C_2 & 0 & 0 \\ C_3\sin\vartheta\cos\varphi & C_4\sin\vartheta\cos\varphi & C_5\cos\vartheta\cos\varphi - C_6\sin\varphi & -\sin\vartheta(C_5\sin\varphi + C_6\cos\vartheta\cos\varphi) \\ C_3\sin\vartheta\sin\varphi & C_4\sin\vartheta\sin\varphi & C_5\cos\vartheta\sin\varphi + C_6\cos\varphi & \sin\vartheta(C_5\cos\varphi - C_6\cos\vartheta\sin\varphi) \\ C_3\cos\vartheta & C_4\cos\vartheta & -C_5\sin\vartheta & C_6\sin^2\vartheta \end{pmatrix},$$

where the six free functions $C_I = C_I(t,r)$ $(I = 1,..6)$ can depend on time and the radial coordinate

Solving antisymmetric field equations

There are two different tetrads which solve the antisymmetric field equation and they have the same metric[3]

$$e_{(1)}{}^{A}{}_{\mu} = \begin{pmatrix} \mathcal{A}(r) & 0 & 0 & 0 \\ 0 & \mathcal{B}(r)\sin\vartheta\cos\varphi & \xi r\cos\vartheta\cos\varphi & -r\xi\sin\vartheta\sin\varphi \\ 0 & \mathcal{B}(r)\sin\vartheta\sin\varphi & \xi r\cos\vartheta\sin\varphi & \xi r\sin\vartheta\cos\varphi \\ 0 & \mathcal{B}(r)\cos\vartheta & -r\xi\sin\vartheta & 0 \end{pmatrix}, \quad \xi = \pm 1,$$

$$e_{(2)}{}^{A}{}_{\mu} = \begin{pmatrix} 0 & i\mathcal{B}(r) & 0 & 0 \\ i\mathcal{A}(r)\sin\vartheta\cos\varphi & 0 & -r\sin\varphi & -r\sin\vartheta\cos\vartheta\cos\varphi \\ i\mathcal{A}(r)\sin\vartheta\sin\varphi & 0 & \chi r\cos\varphi & -r\sin\vartheta\cos\vartheta\sin\varphi \\ i\mathcal{A}(r)\cos\vartheta & 0 & 0 & r\sin^2\vartheta \end{pmatrix},$$

$$\mathrm{d}s^2 = \mathcal{A}(r)^2\mathrm{d}t^2 - \mathcal{B}(r)^2\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2\,.$$

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The phenomenology of these two tetrads will be different!

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Some remarks about the tetrads

- $e_{(1)\mu}^{A}$ has been derived before but different papers did not notice that $\xi = +1$ has a different value of T, B than $\xi = -1$ and then different predictions.

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- For the complex tetrad, all the quantities (torsion tensor, scalars, etc) are real.
- Since we couple matter with the metric, these imaginary terms are not seen, so nothing is wrong with it.

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\mathcal{A}^2(r) &= 1-\frac{2M}{r}+\epsilon\, a(r)\,,\\
\mathcal{B}^2(r) &= \Big(1-\frac{2M}{r}\Big)^{-1}+\epsilon\, b(r)\,,\\
f(T,B) &= T+\epsilon\,\tilde{f}(T,B)=T+\epsilon\,\frac{1}{2}\Big(\alpha T^2+\beta B^2+\gamma BT\Big)\,.
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- We found perturbed solutions with $M\neq 0$ and they contain logarithmic terms.
- Depending on how one sets the integration constants, they can describe (or not) black holes.

Solutions for the complex tetrad

- The symmetric field eqs for $e^{A}_{(2)\mu}$ are simpler. Surprisingly, for any form of f, the metric functions MUST respect

$$\mathcal{B}(r) = \pm \frac{\sqrt{-r^2\mathcal{A}(r)\mathcal{A}''(r) - r^2\mathcal{A}(r)'^2 + \mathcal{A}(r)^2}}{\mathcal{A}(r)}\,.$$

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- We found perturbed solutions around Schwarzschild for any power law $f(T,B)$.
- We also found exact solutions.

Solutions for the complex tetrad - similar RN

- The first interesting black hole one is

$$\mathrm{d}s^2 = \Big(1 - \frac{2M}{r} + \frac{Q}{r^2}\Big)dt^2 - \Big(\frac{2Mr - Q - r^2}{2Q - r^2}\Big)^{-1} dr^2 - r^2 d\Omega^2\,,$$

which looks like Reissner–Nordström but it does not have $g_{tt} = -1/g_{rr}$.

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  which looks like Reissner–Nordström but it does not have $g_{tt} = -1/g_{rr}$.
- This solution has two event horizons and can have any sign for Q.
- The form of the theory is

$$f(T) = 4f_0\,\frac{\Big(2 \pm \sqrt{Q^2T^2 - 2QT + 4}\Big)}{\Big(QT + 2 \pm \sqrt{Q^2T^2 - 2QT + 4}\Big)\sqrt{8 - 2QT \pm 4\sqrt{Q^2T^2 - 2QT + 4}}}\,.$$

Solutions for the complex tetrad - Born-Infield

- A quite well-studied theory inspired by Born-Infield electromagnetism is

$$f(T) = \lambda\left(\sqrt{1+\frac{2T}{\lambda}}-1\right),$$

with λ being the so-called Born-Infeld parameter. It is easy to notice that when $T/\lambda \ll 1$, one obtains $f(T) = T - T^2/(2\lambda) + \mathcal{O}(1/\lambda^2)$.

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- We found an exact black hole solution to this theory

$$\mathrm{d}s^2 = \frac{a_1^2}{r}\Big[\sqrt{\lambda}(a_0\lambda + r) - 2\tan^{-1}\left(\frac{\sqrt{\lambda}r}{2}\right)\Big]\mathrm{d}t^2 - \frac{\lambda^{5/2}r^5}{(4+r^2\lambda)^2}\Big[\sqrt{\lambda}(a_0\lambda + r) - 2\tan^{-1}\left(\frac{\sqrt{\lambda}r}{2}\right)\Big]^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2\,.$$

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- One can set $a_1^2 = 1/\sqrt{\lambda}$ to get asymptotically flatness. Further, if we choose $a_0 = -2M/\lambda$ and expands the metric up to $\mathcal{O}(1/\lambda^2)$, we find

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\begin{aligned}
ds^2 &= \Big[1 - \frac{2M}{r} + \frac{4}{\lambda r^2} - \frac{\pi}{\sqrt{\lambda} r}\Big] dt^2 \\
&\quad - \Big[1 - \frac{2M}{r} - \frac{16M}{\lambda r^3} + \frac{12}{\lambda r^2} - \frac{\pi}{\sqrt{\lambda} r}\Big]^{-1} dr^2 - r^2 d\Omega^2 + \mathcal{O}(1/\lambda^2)\,.
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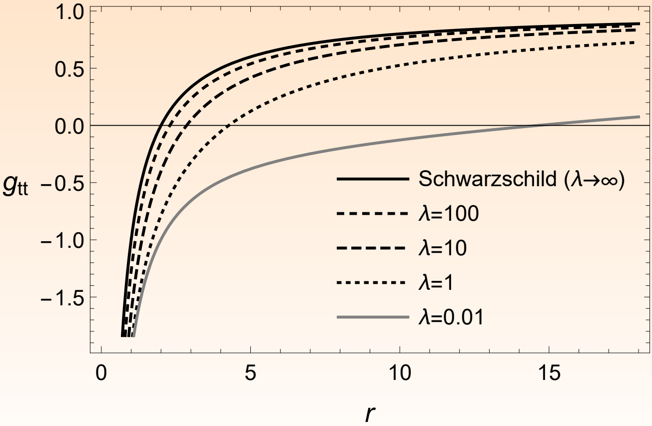
- This is a generalization of a Schwarzschild black hole with one horizon $r_h = 2M + \frac{\pi}{\sqrt{\lambda}} - \frac{2M}{\lambda} + \mathcal{O}(1/\lambda^2)$.

Solutions for the complex tetrad - Born-Infield

We also checked numerically that there is only one horizon:



Theorem spherical symmetry - valid for the two tetrads

In $f(T)$ gravity

In $f(T)$ gravity, only for the case where the model is at most TEGR + Constant, the $\mathcal{A}(r)$ and $\mathcal{B}(r)$ take on the reciprocal of each other ($g_{tt} = -1/g_{rr}$). Moreover, the solution in this case is the Schwarzschild de Sitter solution.

Scalarised black holes in Riemannian geometry

- In GR(+Maxwell), black holes are only characterised by its mass, charge and angular momentum no-hair theorem.

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- Several studies have been carried out for scalar-tensor theories like

$$S = \frac{1}{2\kappa^2} \int_M \left[\mathcal{F}(\psi) \mathring{R} + 2\mathcal{B}(\psi) X - 2\kappa^2 \mathcal{V}(\psi) \right] \sqrt{-g}\, \mathrm{d}^4 x \,.$$

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- One can circumvent the no-hair theorem by having some particular potentials and coupling functions.
- It is well known that spontenous scalarization of black holes occur in other theories like the ones with coupling with Gauss-Bonnet.

Scalar-torsion theories

- One can formulate Teleparallel theories with a scalar field, for example[4]:

$$S=\frac{1}{2\kappa^2}\int_M\Big[-\mathcal{A}(\psi)T-\tilde{\mathcal{C}}(\psi)B+2\mathcal{B}(\psi)X-2\kappa^2\mathcal{V}(\psi)\Big]\sqrt{-g}\,\mathrm{d}^4x\,,$$

where $X=-\frac{1}{2}g^{\mu\nu}\partial_\mu\psi\partial_\nu\psi$.

[4] S. Bahamonde and M. Wright, Phys. Rev. D **92** (2015) no.8, 084034; M. Zubair, S. Bahamonde and M. Jamil, Eur. Phys. J. C **77** (2017) no.7, 472; M. Hohmann and C. Pfeifer, Phys. Rev. D **98** (2018) no.6, 064003.

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  where $X=-\frac{1}{2}g^{\mu\nu}\partial_\mu\psi\partial_\nu\psi$.
- Since $\overset{\circ}{R}=-T+B$, when $\mathcal{A}(\psi)=-\tilde{\mathcal{C}}(\psi)$ the above theory is exactly the same as the standard non-minimally one..

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- The Eqs can be written as

$$\mathring{R}_{\mu\nu} - \frac{1}{2}\mathring{R}g_{\mu\nu} = -\frac{1}{\mathcal{A}(\psi)}\Big[\left(\mathcal{A}'(\psi) + \tilde{\mathcal{C}}'(\psi)\right) S_{(\mu\nu)}{}^{\rho}\psi_{,\rho} + \left(\frac{1}{2}\beta - \tilde{\mathcal{C}}''(\psi)\right)\psi_{,\rho}\psi_{,\sigma}g^{\rho\sigma}g_{\mu\nu} - (\beta - \tilde{\mathcal{C}}''(\psi))\psi_{,\mu}\psi_{,\nu} + \tilde{\mathcal{C}}'(\psi)\left(\mathring{\nabla}_{\mu}\mathring{\nabla}_{\nu}\psi - \mathring{\Box}\psi g_{\mu\nu}\right) + \kappa^2\mathcal{V}(\psi)g_{\mu\nu}\Big] =: \Theta^{(\psi)}{}_{\mu\nu}\,.$$

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$$-(\beta-\tilde{\mathcal{C}}''(\psi))\psi_{,\mu}\psi_{,\nu}+\tilde{\mathcal{C}}'(\psi)\Big(\mathring{\nabla}_{\mu}\mathring{\nabla}_{\nu}\psi-\mathring{\Box}\psi g_{\mu\nu}\Big)+\kappa^{2}\mathcal{V}(\psi)g_{\mu\nu}\Big]=:\Theta^{(\psi)}{}_{\mu\nu}\,.$$

- By solving the antisymmetric eqs, one gets the same two possible tetrads presented before, with the metric now as

$$\mathrm{d}s^2=A^2\mathrm{d}t^2-\frac{C^2}{A^2}\mathrm{d}r^2-r^2\mathrm{d}\Omega^2\,.$$

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- The Eqs can be written as

$$\mathring{R}_{\mu\nu} - \frac{1}{2}\mathring{R}g_{\mu\nu} = -\frac{1}{\mathcal{A}(\psi)}\Big[\left(\mathcal{A}'(\psi)+\tilde{\mathcal{C}}'(\psi)\right)S_{(\mu\nu)}{}^{\rho}\psi_{,\rho} + \left(\frac{1}{2}\beta - \tilde{\mathcal{C}}''(\psi)\right)\psi_{,\rho}\psi_{,\sigma}g^{\rho\sigma}g_{\mu\nu}$$
$$-(\beta - \tilde{\mathcal{C}}''(\psi))\psi_{,\mu}\psi_{,\nu} + \tilde{\mathcal{C}}'(\psi)\left(\mathring{\nabla}_{\mu}\mathring{\nabla}_{\nu}\psi - \mathring{\Box}\psi g_{\mu\nu}\right) + \kappa^2\mathcal{V}(\psi)g_{\mu\nu}\Big] =: \Theta^{(\psi)}{}_{\mu\nu}\,.$$

- By solving the antisymmetric eqs, one gets the same two possible tetrads presented before, with the metric now as

$$\mathrm{d}s^2 = A^2\mathrm{d}t^2 - \frac{C^2}{A^2}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2\,.$$

- We can express the energy of the scalar field in the black hole exterior region as

$$\mathcal{E}_{\psi} = \int_{r_h}^{\infty}\int_{S^2}\Theta^{(\psi)t}{}_{t}Cr^2\mathrm{d}r\,\mathrm{d}\Omega = 4\pi\int_{r_h}^{\infty}\rho_{\psi}(r)Cr^2\mathrm{d}r\,,$$

where $\rho_{\psi}(r) = \Theta^{(\psi)t}{}_{t}(r)$ will be interpreted as the (effective) scalar field energy density.

$\mathcal{A}(\psi) = \alpha$ only non-minimal coupling between the scalar field and the boundary

- This theory has a GR term plus and a $\tilde{\mathcal{C}}(\psi)B$ term.

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- For the complex tetrad, we found Schwarzschild-de-Sitter solutions with different couplings and potentials.
- We also found a non-asymptotically flat solution

$$
\begin{aligned}
\mathrm{d}s^2 &= r^{-2}\Big(A_0 r^{\pm\sqrt{2w}}+r^{\pm\sqrt{w/2}}\Big)\mathrm{d}t^2 \\
&\quad -\frac{A_0\sqrt{w}}{2}r^{\pm\sqrt{w/2}}\Big(A_0 r^{\pm\sqrt{w/2}}+1\Big)^{-1}(\pm 2\sqrt{2}-\sqrt{w})\mathrm{d}r^2-r^2\mathrm{d}\Omega^2\,,
\end{aligned}
$$

with $w=2-\beta\psi_0^2$ and

$$
\tilde{\mathcal{C}}(\psi)=\Big(\frac{-2\pm\sqrt{2w}}{\psi_0}\Big)\psi\,,\quad \mathcal{V}(\psi)=0\,,\quad \psi=\psi_0\log(r)\,.
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- One interesting one is

$$\mathrm{d}s^2=\left(1-\frac{K}{r}\right)^2\mathrm{d}t^2-\left(1-\frac{K}{r}\right)^{-2}\mathrm{d}r^2-r^2\mathrm{d}\Omega^2\,,$$

with $\psi(r)=-\frac{2\psi_0\sqrt{r}}{K\sqrt{r-K}},\mathcal{A}(\psi)=-\frac{1}{8}\beta\psi^2\,,\mathcal{V}(\psi)=0$.

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  with $\psi(r)=-\frac{2\psi_0\sqrt{r}}{K\sqrt{r-K}},\mathcal{A}(\psi)=-\frac{1}{8}\beta\psi^2\,,\mathcal{V}(\psi)=0$.
- The metric is the same as the Bocharova–Bronnikov–Melnikov–Bekenstein (BBMB) solution found in Riemannian conformal scalar-vacuum theory!

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- Another interesting scalarised black hole solution found is

$$\begin{aligned}\mathrm{d}s^2 &= \Big(2-\frac{r}{2K}+\frac{\sqrt{r(r-4K)}}{2K}\Big)^2\mathrm{d}t^2\\ &\quad -\Big(2-\frac{r}{2K}+\frac{\sqrt{r(r-4K)}}{2K}\Big)^{-2}\mathrm{d}r^2-r^2\mathrm{d}\Omega^2\,,\end{aligned}$$

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- If $K \ll 1$ the metric components behave as

$$g_{tt} = -1/g_{rr} = 1 - \frac{2K}{r} - \frac{3K^2}{r^2} + \mathcal{O}(K^3)\,,$$

which looks like a Reissner-Norstrdom BH with imaginary charge ($Q^2 = -3M^2$).

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- Another exact BH solution (non-asymptotically flat) that we found is

$$\mathrm{d}s^2 = \Big(A_0 - 2Mr^{-2p-1}\Big)\mathrm{d}t^2 - \Big(\frac{A_0(2p+1)}{A_0 - 2Mr^{-2p-1}}\Big)\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2\,,$$

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- This expanded form of the metric contains a logarithmic term as in $f(T)$

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4. $\mathcal{A}=\alpha$, $\tilde{\mathcal{C}}=\frac{\gamma}{m+1}\psi^{m+1}$ and $\frac{1}{\beta}\left(\psi\mathcal{V}'-(m+1)\gamma\psi^m T^r\psi'\right)\leq 0$ or $\frac{(m+1)}{m-1}\frac{1}{\beta}\left(\alpha\mathring{R}+\kappa^2(\psi\mathcal{V}'-4\mathcal{V})\right)\leq 0$.

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- Are these solutions stable? what is the phenomenology of them?