Black holes in Teleparallel gravity

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Gravity Current challenges in black hole physics and cosmology,
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arXiv:2206.02725 and our review arXiv:2106.13793

Outline

Fundamental variables and characteristic tensors

- In the most general metric-affine setting, the fundamental variables are a **metric** $g_{\mu\nu}$ (10 comp.) as well as the coefficients $\tilde{\Gamma}^{\rho}{}_{\mu\nu}$ (64 comp.) of an **affine connection.**

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Torsion	$\tilde{T}^{\mu}{}_{\nu\rho} = \tilde{\Gamma}^{\mu}{}_{\rho\nu} - \tilde{\Gamma}^{\mu}{}_{\nu\rho}$
Nonmetricity	$\tilde{Q}_{\mu\nu\rho} = \tilde{\nabla}_{\mu}g_{\nu\rho} = \partial_{\mu}g_{\nu\rho} - \tilde{\Gamma}^{\sigma}{}_{\nu\mu}g_{\sigma\rho} - \tilde{\Gamma}^{\sigma}{}_{\rho\mu}g_{\nu\sigma}$

Tetrads and spin connection

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where η_{ab} is the Minkowski metric.

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- The curvature becomes

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$$\tilde{R}=\mathring{R}+\Big(T+2\mathring{\nabla}_{\mu}(\sqrt{-g}T^{\rho}{}_{\rho}{}^{\mu})\Big)+\Big(Q+\mathring{\nabla}_{\mu}Q^{\mu\nu}{}_{\nu}-\mathring{\nabla}_{\nu}Q_{\mu}{}^{\mu\nu}\Big)+C$$

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with

$$T:=T^{\rho\lambda\kappa}T_{\rho\lambda\kappa}+2T^{\rho\lambda\kappa}T_{\kappa\rho\lambda}-4T_{\rho}{}^{\kappa}{}_{\kappa}T^{\rho\lambda}{}_{\lambda}\,,\quad \text{Torsion scalar}\,,$$
$$Q:=-\frac{1}{4}Q_{\alpha\beta\gamma}Q^{\alpha\beta\gamma}+\frac{1}{2}Q_{\alpha\beta\gamma}Q^{\beta\alpha\gamma}+\frac{1}{4}Q_{\alpha}Q^{\alpha}-\frac{1}{2}Q_{\alpha}\bar{Q}^{\alpha}\,,\ \text{Nonmetricity scalar}\,,$$
$$C:=2(Q_{\kappa\rho\lambda}T^{\lambda\kappa\rho}+Q_{\rho}{}^{\sigma}{}_{\sigma}T^{\rho\kappa}{}_{\kappa}-Q^{\sigma}{}_{\sigma\rho}T^{\rho\kappa}{}_{\kappa})\,.$$

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where $\kappa^2 = 8\pi G$ and $L_{\rm m}$ is any matter Lagrangian.

- The Einstein's field equations are obtained by taking variations w/r to the metric: $\boxed{\mathring{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathring{R} = \kappa^2 T_{\mu\nu}}$.

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(torsional) Teleparallel equivalent of GR (TEGR) action

$$S_{\rm TEGR} = \int \Big[-\frac{1}{2\kappa^2}T + L_{\rm m}\Big]\, e\, d^4x\,.$$

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- Since $\mathring{R}$ differs by T by a boundary term B, **the equations of TEGR are equivalent to the Einstein's field equations.**

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- Since $\mathring{R}$ differs by Q by a boundary term B_Q, **the equations of STEGR are equivalent to the GR eqs.**

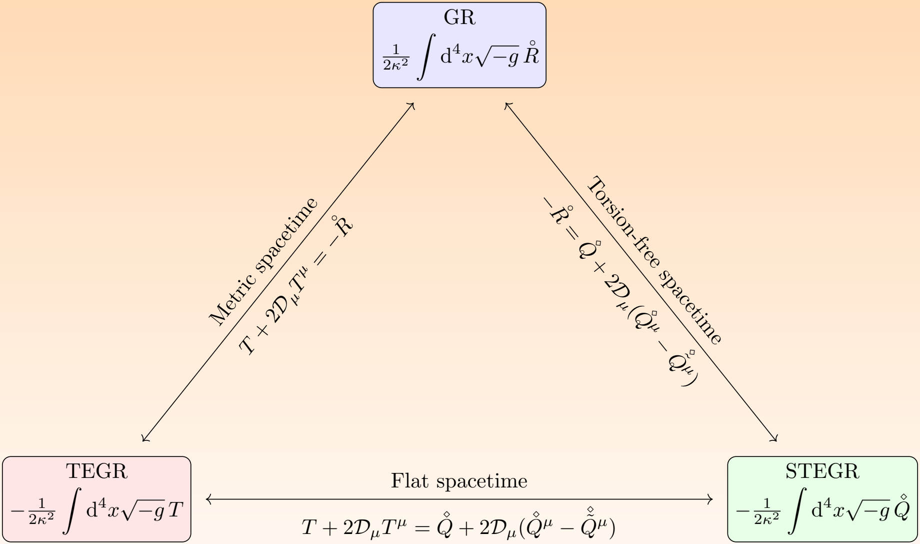

Figure: Geometrical trinity of gravity (S. Bahamonde et.al., "Teleparallel Gravity: From Theory to Cosmology," [arXiv:2106.13793 [gr-qc]].; J. Beltrán Jiménez, L. Heisenberg and T. S. Koivisto, "The Geometrical Trinity of Gravity," Universe **5** (2019) no.7, 173.)

Modified teleparallel theories

What happens if we modify TEGR and STEGR?

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Are torsional Teleparallel theories Lorentz invariant?

There is a lot of misconceptions in the literature since TEGR and their first modifications were proposed in the pure-tetrad formalism, which assumes that ${\omega^a}_{b\mu} = 0$ globally.

Modified teleparallel theories

Is it possible to formulate Teleparallel theories in a fully invariant form?

Yes, and the way to do this is by incorporating the spin connection ${\omega^a}_{b\mu}$ in the formulation. If one introduces this quantity, the torsion tensor is always fully covariant (covariant under both diffeo and local Lorentz) and then, any action constructed from it will be fully invariant.

See[1]

[1] M. Krššák and E. N. Saridakis, Class. Quant. Grav. **33** (2016) no.11, 115009; M. Krssak, R. J. van den Hoogen, J. G. Pereira, C. G. Böhmer and A. A. Coley, Class. Quant. Grav. **36** (2019) no.18, 183001.

Theories with scalar torsion and boundary term - $f(T, B)$

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- There has been quite a lot of study about this theory in the last years.

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Field equations and spherical symmetry

- The separate tetrad and spin connection variations produce the field equations

$$W_{(\mu\nu)} = \kappa^2 \Theta_{\mu\nu}\,,$$
$$W_{[\mu\nu]} = \Big[(\partial_\rho f_B) + (\partial_\rho f_T)\Big] S_{[\mu}{}^{\rho}{}_{\nu]} \propto T^{\rho}{}_{[\mu\nu}\partial_{\rho]}(f_T + f_B) = 0\,.$$

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- The three equations of motion obtained are not fully independent of each other.
- The antisymmetric field equation coincides with the spin connection equation.

Field equations and spherical symmetry

Killing eqs:

$$\mathcal{L}_{Z_\zeta} e^A{}_\mu = -\lambda_\zeta^A{}_B e^B{}_\mu\,, \quad \mathcal{L}_{Z_\zeta}\omega^A{}_{B\mu} = \partial_\mu \lambda_\zeta^A{}_B + \omega^A{}_{C\mu}\lambda_\zeta^C{}_B - \omega^C{}_{B\mu}\lambda_\zeta^A{}_C\,.$$

[3] M. Hohmann, L. Järv, M. Krššák and C. Pfeifer, Phys. Rev. D **100** (2019) no.8, 084002

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By solving these eqs in the Weitzenböck gauge we find[3]

$$e^A{}_\nu = \begin{pmatrix} C_1 & C_2 & 0 & 0 \\ C_3\sin\vartheta\cos\varphi & C_4\sin\vartheta\cos\varphi & C_5\cos\vartheta\cos\varphi - C_6\sin\varphi & -\sin\vartheta(C_5\sin\varphi + C_6\cos\vartheta\cos\varphi) \\ C_3\sin\vartheta\sin\varphi & C_4\sin\vartheta\sin\varphi & C_5\cos\vartheta\sin\varphi + C_6\cos\varphi & \sin\vartheta(C_5\cos\varphi - C_6\cos\vartheta\sin\varphi) \\ C_3\cos\vartheta & C_4\cos\vartheta & -C_5\sin\vartheta & C_6\sin^2\vartheta \end{pmatrix},$$

where the six free functions $C_I = C_I(t,r)$ $(I = 1, ..6)$ can depend on time and the radial coordinate

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Solving antisymmetric field equations

There are two different tetrads which solve the antisymmetric field equation and they have the same metric[4]

$$
e^A_{(1)\,\mu} = \begin{pmatrix} \mathcal{A}(r) & 0 & 0 & 0 \\ 0 & \mathcal{B}(r)\sin\vartheta\cos\varphi & \xi r\cos\vartheta\cos\varphi & -r\xi\sin\vartheta\sin\varphi \\ 0 & \mathcal{B}(r)\sin\vartheta\sin\varphi & \xi r\cos\vartheta\sin\varphi & \xi r\sin\vartheta\cos\varphi \\ 0 & \mathcal{B}(r)\cos\vartheta & -r\xi\sin\vartheta & 0 \end{pmatrix}, \quad \xi = \pm 1\,,
$$

$$
e^A_{(2)\,\mu} = \begin{pmatrix} 0 & i\mathcal{B}(r) & 0 & 0 \\ i\mathcal{A}(r)\sin\vartheta\cos\varphi & 0 & -r\sin\varphi & -r\sin\vartheta\cos\vartheta\cos\varphi \\ i\mathcal{A}(r)\sin\vartheta\sin\varphi & 0 & \chi r\cos\varphi & -r\sin\vartheta\cos\vartheta\sin\varphi \\ i\mathcal{A}(r)\cos\vartheta & 0 & 0 & r\sin^2\vartheta \end{pmatrix},
$$

$$
\mathrm{d}s^2 = \mathcal{A}(r)^2\mathrm{d}t^2 - \mathcal{B}(r)^2\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2\,.
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[4] S. Bahamonde, A. Golovnev, M. J. Guzmán, J. L. Said and C. Pfeifer, JCAP **01** (2022) no.01, 037

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$$\mathrm{d}s^2=\mathcal{A}(r)^2\mathrm{d}t^2-\mathcal{B}(r)^2\mathrm{d}r^2-r^2\mathrm{d}\Omega^2\,.$$

The phenomenology of these two tetrads will be different! We found exact solutions for different f.

[4] S. Bahamonde, A. Golovnev, M. J. Guzmán, J. L. Said and C. Pfeifer, JCAP **01** (2022) no.01, 037

Solution for the complex tetrad - Born-Infield

- A quite well-studied theory inspired by Born-Infield electromagnetism is

$$f(T) = \lambda\left(\sqrt{1+\frac{2T}{\lambda}} - 1\right),$$

with λ being the so-called Born-Infeld parameter. It is easy to notice that when $T/\lambda \ll 1$, one obtains $f(T) = T - T^2/(2\lambda) + \mathcal{O}(1/\lambda^2)$.

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- We found an exact black hole solution to this theory(JCAP **01** (2022) no.01, 0370)

$$\begin{aligned}\mathrm{d}s^2 &= \frac{a_1^2}{r}\Big[\sqrt{\lambda}(a_0\lambda+r)-2\tan^{-1}\Big(\frac{\sqrt{\lambda}r}{2}\Big)\Big]\mathrm{d}t^2\\ &-\frac{\lambda^{5/2}r^5}{(4+r^2\lambda)^2}\Big[\sqrt{\lambda}(a_0\lambda+r)-2\tan^{-1}\Big(\frac{\sqrt{\lambda}r}{2}\Big)\Big]^{-1}\mathrm{d}r^2-r^2\mathrm{d}\Omega^2\,. \qquad (2)\end{aligned}$$

Solutions for the complex tetrad - Born-Infield

One can set $a_1^2 = 1/\sqrt{\lambda}$ to get asymptotically flatness. Further, if we choose $a_0 = -2M/\lambda$ and expands the metric up to $\mathcal{O}(1/\lambda^2)$, we find

$$ds^2 = \Big[1 - \frac{2M}{r} + \frac{4}{\lambda r^2} - \frac{\pi}{\sqrt{\lambda} r}\Big] dt^2 - \Big[1 - \frac{2M}{r} - \frac{16M}{\lambda r^3} + \frac{12}{\lambda r^2} - \frac{\pi}{\sqrt{\lambda} r}\Big]^{-1} dr^2 - r^2 d\Omega^2 + \mathcal{O}(1/\lambda^2)\,.$$

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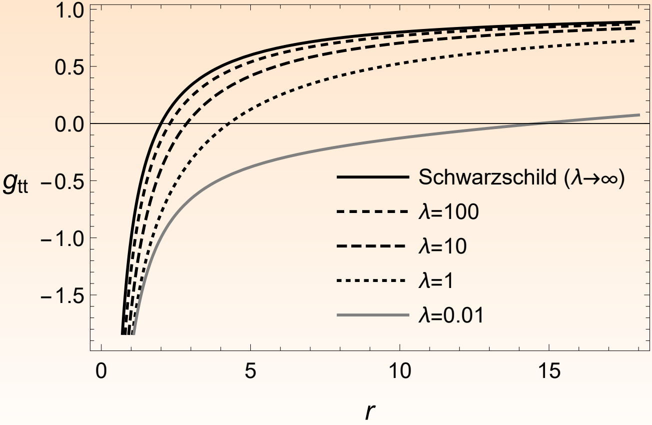
This is a generalization of a Schwarzschild black hole with one horizon $r_h = 2M + \frac{\pi}{\sqrt{\lambda}} - \frac{2M}{\lambda} + \mathcal{O}(1/\lambda^2)$.

Solutions for the complex tetrad - Born-Infield

We also checked numerically that there is only one horizon:



Theorem spherical symmetry - valid for the two tetrads

Theorem for $f(T)$ gravity

In $f(T)$ gravity, only for the case where the model is at most TEGR + Constant, the $\mathcal{A}(r)$ and $\mathcal{B}(r)$ take on the reciprocal of each other ($g_{tt} = -1/g_{rr}$). Moreover, the solution in this case is the Schwarzschild de Sitter solution.

Scalarised black holes in Riemannian geometry

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- Several studies have been carried out for scalar-tensor theories like

$$S = \frac{1}{2\kappa^2}\int_M \left[\mathcal{F}(\psi)\mathring{R} + 2\mathcal{B}(\psi)X - 2\kappa^2\mathcal{V}(\psi)\right]\sqrt{-g}\,\mathrm{d}^4x\,.$$

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- When $\mathcal{F}(\psi) = 1$ and $V \propto -\psi^2$, there is a no-hair theorem.
- One can circumvent the no-hair theorem by having some particular potentials and coupling functions.
- It is well known that spontenous scalarization of black holes occur in other theories like the ones with coupling with Gauss-Bonnet.

Scalar-torsion theories

- One can formulate Teleparallel theories with a scalar field, for example[5]:

$$S=\frac{1}{2\kappa^2}\int_M\left[-\mathcal{A}(\psi)T-\tilde{\mathcal{C}}(\psi)B+2\mathcal{B}(\psi)X-2\kappa^2\mathcal{V}(\psi)\right]\sqrt{-g}\,\mathrm{d}^4x\,,$$

  where $X=-\frac{1}{2}g^{\mu\nu}\partial_\mu\psi\partial_\nu\psi$.

[5] S. Bahamonde and M. Wright, Phys. Rev. D **92** (2015) no.8, 084034; M. Zubair, S. Bahamonde and M. Jamil, Eur. Phys. J. C **77** (2017) no.7, 472; M. Hohmann and C. Pfeifer, Phys. Rev. D **98** (2018) no.6, 064003.

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where $X = -\frac{1}{2}g^{\mu\nu}\partial_\mu\psi\partial_\nu\psi$.

- Since $\mathring{R} = -T + B$, when $\mathcal{A}(\psi) = -\tilde{\mathcal{C}}(\psi)$ the above theory is exactly the same as the standard non-minimally one..

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- This theory has a coupling like $\mathcal{A}(\psi)T$.

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- For this case, we found exact solutions for the real and complex tetrad[6].

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$$\mathrm{d}s^2 = \left(1 - \frac{K}{r}\right)^2 \mathrm{d}t^2 - \left(1 - \frac{K}{r}\right)^{-2} \mathrm{d}r^2 - r^2 \mathrm{d}\Omega^2 \,,$$

with $\psi(r) = -\frac{2\psi_0\sqrt{r}}{K\sqrt{r-K}}, \mathcal{A}(\psi) = -\frac{1}{8}\beta\psi^2 \,, \mathcal{V}(\psi) = 0$.

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- The metric is the same as the Bocharova–Bronnikov–Melnikov–Bekenstein (BBMB) solution found in Riemannian conformal scalar-vacuum theory!

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$\tilde{\mathcal{C}}(\psi)=0$ only non-minimal coupling between the scalar field and the torsion scalar

- Another interesting scalarised black hole solution found is

$$\begin{aligned} \mathrm{d}s^2 &= \Big(2-\frac{r}{2K}+\frac{\sqrt{r(r-4K)}}{2K}\Big)^2\mathrm{d}t^2 \\ &\quad -\Big(2-\frac{r}{2K}+\frac{\sqrt{r(r-4K)}}{2K}\Big)^{-2}\mathrm{d}r^2-r^2\mathrm{d}\Omega^2\,, \end{aligned}$$

where the scalar field is non-trivial and $\mathcal{A}(\psi)=\frac{3\beta}{8}\psi^2$. This metric is asymptotically flat.

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  where the scalar field is non-trivial and $\mathcal{A}(\psi) = \frac{3\beta}{8}\psi^2$. This metric is asymptotically flat.
- If $K \ll 1$ the metric components behave as

$$g_{tt} = -1/g_{rr} = 1 - \frac{2K}{r} - \frac{3K^2}{r^2} + \mathcal{O}(K^3)\,,$$

  which looks like a Reissner-Norstrdom BH with imaginary charge ($Q^2 = -3M^2$).

Scalar-tensor theories in symmetric TG

- Similarly as with the torsional case, one can formulate analogues theories in symmetric TG:[7]

$$S=\frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g}\,\Big(\mathcal{A}(\Phi)Q-\mathcal{B}(\Phi)g^{\alpha\beta}\partial_\alpha\Phi\partial_\beta\Phi-2\mathcal{V}(\Phi)\Big)\,. \tag{3}$$

where $Q\equiv-\frac{1}{4}\,Q_{\lambda\mu\nu}Q^{\lambda\mu\nu}+\frac{1}{2}\,Q_{\lambda\mu\nu}Q^{\mu\nu\lambda}+\frac{1}{4}\,Q_\mu Q^\mu-\frac{1}{2}\,Q_\mu\hat{Q}^\mu$.

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- If $\mathcal{A}(\Phi)=const$, we just have Einstein-Klein-Gordon. Also, the above theory contains $f(Q)$ gravity.
- In our recent paper, we analysed spherical symmetry to this theory and found several exact black hole solutions.

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Scalar-tensor theories in symmetric TG

- Solving the conditions $\mathcal{L}_{Z_\zeta} g_{\mu\nu} = 0\,,\mathcal{L}_{Z_\zeta}\Gamma^{\lambda}{}_{\mu\nu} = 0$, we find that there are two different connections satisfying spherical symmetry (respecting zero curvature and zero torsion)[8].

[8] F. D'Ambrosio, S. D. B. Fell, L. Heisenberg and S. Kuhn, Phys. Rev. D **105** (2022) no.2, 024042

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- The second set (2nd connection) has non-trivial black hole solutions and they are different to the Riemannian case. They can be split into two subcases.
- In our paper[9], we showed that the theorem mentioned for $f(T)$ gravity, is also valid for $f(Q)$ gravity! so that $g_{rr} \neq -1/g_{tt}$ to go beyond GR.

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- We found several scalarized BH solutions[10] for our theory which are new ones. Mainly without the potential and with a coupling like $Q\Phi^2$.

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- The BBHB is also a solution of the symmetric scalar-tensor theory.
- An interesting new solution in this sector:

$$ds^2 = -\left(1+W\left(\frac{-M}{r}\right)\right)^2 \mathrm{d}t^2 + \left(1+W\left(\frac{-M}{r}\right)\right)^{-2} \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2\,, \quad (4)$$

$$\Phi(r) = \Phi_0 \left(-\frac{M}{rW\left(-\frac{M}{r}\right)}\right)^{1/2}\,, \quad \mathcal{A}(\Phi) = \frac{\beta\Phi^2}{8}\,, \quad \mathcal{V}(\Phi) = 0\,, \quad (5)$$

where $W(z)$ is the Lambert function defined as $W(z)e^{W(z)} = z$ and M represents the ADM mass of the black hole.

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Conclusions

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- We found that the first exact non-trivial Black hole solutions in modified teleparallel gravity, for $f(T)$ gravity and also we found scalarised black hole solutions for scalar-torsion theories and symmetric non-metricity scalar-tensor theories.

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- We found that the first exact non-trivial Black hole solutions in modified teleparallel gravity, for $f(T)$ gravity and also we found scalarised black hole solutions for scalar-torsion theories and symmetric non-metricity scalar-tensor theories.
- We formulated no-hair theorems for scalar-torsion gravity and symmetric (non-metricity) scalar-tensor theories.

Conclusions and final remarks

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