# Algebraic classification of the gravitational field in Weyl-Cartan space-times

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# Outline

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## Riemmanian Curvature decomposition

$$R_{\lambda\rho\mu\nu} = W_{\lambda\rho\mu\nu} + \frac{1}{2}\Big(g_{\lambda\mu} \not{R}_{\rho\nu} + g_{\rho\nu} \not{R}_{\lambda\mu} - g_{\lambda\nu} \not{R}_{\rho\mu} - g_{\rho\mu} \not{R}_{\lambda\nu}\Big) + \frac{1}{6} R\, g_{\lambda[\mu} g_{\nu]\rho}\,,$$

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- **Result in Riemannian geometry:** Weyl has 6 types **(Petrov classification)**; Ricci traceless has 15 types**(Segre classification)**;
- **What happens in GR in vacuum?** $\not{R}_{\rho\nu} = R = 0$ and then the curvature is fully characterised by the Weyl tensor with their 6 types.

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- *Goldberg-Sachs theorem:* A vacuum solution of the Einstein's field equations admits a shear-free null geodesic congruence if and only if the conformal part of the Riemann tensor is algebraically special.

# Fundamental variables and characteristic tensors

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|---|---|
| **Curvature** | $\tilde{R}^{\mu}{}_{\nu\rho\sigma} = \partial_{\rho}\tilde{\Gamma}^{\mu}{}_{\nu\sigma} - \partial_{\sigma}\tilde{\Gamma}^{\mu}{}_{\nu\rho} + \tilde{\Gamma}^{\mu}{}_{\tau\rho}\tilde{\Gamma}^{\tau}{}_{\nu\sigma} - \tilde{\Gamma}^{\mu}{}_{\tau\sigma}\tilde{\Gamma}^{\tau}{}_{\nu\rho}$ |
| **Torsion** | $\tilde{T}^{\mu}{}_{\nu\rho} = \tilde{\Gamma}^{\mu}{}_{\rho\nu} - \tilde{\Gamma}^{\mu}{}_{\nu\rho}$ |
| **Nonmetricity** | $\tilde{Q}_{\mu\nu\rho} = \tilde{\nabla}_{\mu}g_{\nu\rho} = \partial_{\mu}g_{\nu\rho} - \tilde{\Gamma}^{\sigma}{}_{\nu\mu}g_{\sigma\rho} - \tilde{\Gamma}^{\sigma}{}_{\rho\mu}g_{\nu\sigma}$ |

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- Three independent second order tensors defined from the first contractions of the curvature tensor:

$$\tilde{R}_{\mu\nu} = \tilde{R}^{\lambda}{}_{\mu\lambda\nu}\,, \quad \hat{R}_{\mu\nu} = \tilde{R}_{\mu}{}^{\lambda}{}_{\nu\lambda}\,, \quad \tilde{R}^{\lambda}{}_{\lambda\mu\nu} = \nabla_{[\nu} Q_{\mu]\lambda}{}^{\lambda}\,. \tag{6}$$

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- In 4D, $\tilde{R}_{\rho\sigma\mu\nu}$ has $\#96$; $T_{\rho\sigma\mu}$ has $\#24$; $Q_{\rho\sigma\mu}$ has $\#40$.

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- Possible new effects such as gravitational spin-orbit interaction could be obtained in the decoupling limit.
- What can we do to obtain such axially symmetric solutions? One possible route is to impose additional symmetries for our field strengths tensors using an algebraic classification

# Curvature decomposition in Metric-Affine geometry

- First, we need to find the building blocks of the general curvature tensor (recall that in Riemannan geometry we had 3: $W_{\alpha\beta\mu\nu}, \tilde{\not{R}}_{\mu\nu}, R$).

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**General curvature**

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- We find that there are 11 building blocks:

| Building block | Number of independent components | Limit in Riemannian geometry |
|---|---|---|
| ${}^{(1)}\tilde{Z}_{\lambda\rho\mu\nu}$ | 30 | zero |
| ${}^{(1)}\tilde{W}_{\lambda\rho\mu\nu}$ | 10 | Weyl tensor $W_{\lambda\rho\mu\nu}$ |
| $\tilde{\not{R}}^{(T)}_{\lambda[\rho\mu\nu]}$ | 9 | zero |
| $\tilde{\not{R}}^{(Q)}_{\lambda[\rho\mu\nu]}$ | 9 | zero |
| $\tilde{\not{R}}_{(\mu\nu)}$ | 9 | Ricci traceless $\not{R}_{\mu\nu}$ |
| $\hat{\not{R}}^{(Q)}_{(\mu\nu)}$ | 9 | zero |
| $\tilde{R}^{(T)}_{[\mu\nu]}$ | 6 | zero |
| $\hat{R}^{(Q)}_{[\mu\nu]}$ | 6 | zero |
| $\tilde{R}^{\lambda}{}_{\lambda\mu\nu}$ | 6 | zero |
| $\tilde{R}$ | 1 | Ricci scalar $R$ |
| $*\tilde{R}$ | 1 | zero |

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- Antisymmetric components
$\tilde{R}_{[\lambda\rho]\mu\nu} = \tilde{W}_{\lambda\rho\mu\nu} = {}^{(1)}\tilde{W}_{\lambda\rho\mu\nu} + {}^{(2)}\tilde{W}_{\lambda\rho\mu\nu} + {}^{(3)}\tilde{W}_{\lambda\rho\mu\nu} + {}^{(4)}\tilde{W}_{\lambda\rho\mu\nu} + {}^{(5)}\tilde{W}_{\lambda\rho\mu\nu} + {}^{(6)}\tilde{W}_{\lambda\rho\mu\nu}$:

$$ {}^{(1)}\tilde{W}_{\lambda\rho\mu\nu} = \tilde{W}_{\lambda\rho\mu\nu} - \sum_{i=2}^{6} {}^{(i)}\tilde{W}_{\lambda\rho\mu\nu} \,, \tag{9} $$

$$ {}^{(2)}\tilde{W}_{\lambda\rho\mu\nu} = \frac{3}{4}\left(\tilde{\not{R}}^{(T)}_{\lambda[\rho\mu\nu]} + \tilde{\not{R}}^{(T)}_{\nu[\lambda\rho\mu]} - \tilde{\not{R}}^{(T)}_{\rho[\lambda\mu\nu]} - \tilde{\not{R}}^{(T)}_{\mu[\lambda\rho\nu]}\right) + \frac{1}{2}\left(\tilde{\not{R}}^{(Q)}_{\mu[\lambda\rho\nu]} - \tilde{\not{R}}^{(Q)}_{\nu[\lambda\rho\mu]}\right), \tag{10} $$

$$ {}^{(3)}\tilde{W}_{\lambda\rho\mu\nu} = -\frac{1}{24} * \tilde{R}\, \varepsilon_{\lambda\rho\mu\nu} \,, \quad {}^{(6)}\tilde{W}_{\lambda\rho\mu\nu} = \frac{1}{6}\tilde{R}\, g_{\lambda[\mu} g_{\nu]\rho} \,, \tag{11} $$

$$ \begin{aligned} {}^{(4)}\tilde{W}_{\lambda\rho\mu\nu} = \frac{1}{4}\Big[ & g_{\lambda\mu}\left(2\tilde{\not{R}}_{(\rho\nu)} + \hat{\not{R}}^{(Q)}_{(\rho\nu)}\right) + g_{\rho\nu}\left(2\tilde{\not{R}}_{(\lambda\mu)} + \hat{\not{R}}^{(Q)}_{(\lambda\mu)}\right) \\ & - g_{\lambda\nu}\left(2\tilde{\not{R}}_{(\rho\mu)} + \hat{\not{R}}^{(Q)}_{(\rho\mu)}\right) - g_{\rho\mu}\left(2\tilde{\not{R}}_{(\lambda\nu)} + \hat{\not{R}}^{(Q)}_{(\lambda\nu)}\right)\Big], \end{aligned} \tag{12} $$

$$ \begin{aligned} {}^{(5)}\tilde{W}_{\lambda\rho\mu\nu} = \frac{1}{4}\Big[ & g_{\lambda\mu}\left(2\tilde{R}^{(T)}_{[\rho\nu]} + \hat{R}^{(Q)}_{[\rho\nu]}\right) + g_{\rho\nu}\left(2\tilde{R}^{(T)}_{[\lambda\mu]} + \hat{R}^{(Q)}_{[\lambda\mu]}\right) + \tilde{R}^{\sigma}{}_{\sigma\lambda[\mu} g_{\nu]\rho} \\ & - g_{\lambda\nu}\left(2\tilde{R}^{(T)}_{[\rho\mu]} + \hat{R}^{(Q)}_{[\rho\mu]}\right) - g_{\rho\mu}\left(2\tilde{R}^{(T)}_{[\lambda\nu]} + \hat{R}^{(Q)}_{[\lambda\nu]}\right) - \tilde{R}^{\sigma}{}_{\sigma\rho[\mu} g_{\nu]\lambda}\Big]. \end{aligned} $$

# Curvature decomposition in Metric-Affine geometry

- Using the 11 building blocks, one can express the curvature tensor into its irreducible decomposition.
- Antisymmetric components
$\tilde{R}_{[\lambda\rho]\mu\nu} = \tilde{W}_{\lambda\rho\mu\nu} = {}^{(1)}\tilde{W}_{\lambda\rho\mu\nu} + {}^{(2)}\tilde{W}_{\lambda\rho\mu\nu} + {}^{(3)}\tilde{W}_{\lambda\rho\mu\nu} + {}^{(4)}\tilde{W}_{\lambda\rho\mu\nu} + {}^{(5)}\tilde{W}_{\lambda\rho\mu\nu} + {}^{(6)}\tilde{W}_{\lambda\rho\mu\nu}$:

$$ {}^{(1)}\tilde{W}_{\lambda\rho\mu\nu} = \tilde{W}_{\lambda\rho\mu\nu} - \sum_{i=2}^{6} {}^{(i)}\tilde{W}_{\lambda\rho\mu\nu} \,, \tag{9} $$

$$ {}^{(2)}\tilde{W}_{\lambda\rho\mu\nu} = \frac{3}{4}\left(\tilde{\not{R}}^{(T)}_{\lambda[\rho\mu\nu]} + \tilde{\not{R}}^{(T)}_{\nu[\lambda\rho\mu]} - \tilde{\not{R}}^{(T)}_{\rho[\lambda\mu\nu]} - \tilde{\not{R}}^{(T)}_{\mu[\lambda\rho\nu]}\right) + \frac{1}{2}\left(\tilde{\not{R}}^{(Q)}_{\mu[\lambda\rho\nu]} - \tilde{\not{R}}^{(Q)}_{\nu[\lambda\rho\mu]}\right) , \tag{10} $$

$$ {}^{(3)}\tilde{W}_{\lambda\rho\mu\nu} = -\frac{1}{24} *\tilde{R}\,\varepsilon_{\lambda\rho\mu\nu} \,, \quad {}^{(6)}\tilde{W}_{\lambda\rho\mu\nu} = \frac{1}{6}\tilde{R}\,g_{\lambda[\mu}g_{\nu]\rho} \,, \tag{11} $$

$$ \begin{aligned} {}^{(4)}\tilde{W}_{\lambda\rho\mu\nu} = \frac{1}{4}\Big[ & g_{\lambda\mu}\left(2\tilde{\not{R}}_{(\rho\nu)} + \hat{\not{R}}^{(Q)}_{(\rho\nu)}\right) + g_{\rho\nu}\left(2\tilde{\not{R}}_{(\lambda\mu)} + \hat{\not{R}}^{(Q)}_{(\lambda\mu)}\right) \\ & - g_{\lambda\nu}\left(2\tilde{\not{R}}_{(\rho\mu)} + \hat{\not{R}}^{(Q)}_{(\rho\mu)}\right) - g_{\rho\mu}\left(2\tilde{\not{R}}_{(\lambda\nu)} + \hat{\not{R}}^{(Q)}_{(\lambda\nu)}\right)\Big] , \end{aligned} \tag{12} $$

$$ \begin{aligned} {}^{(5)}\tilde{W}_{\lambda\rho\mu\nu} = \frac{1}{4}\Big[ & g_{\lambda\mu}\left(2\tilde{R}^{(T)}_{[\rho\nu]} + \hat{R}^{(Q)}_{[\rho\nu]}\right) + g_{\rho\nu}\left(2\tilde{R}^{(T)}_{[\lambda\mu]} + \hat{R}^{(Q)}_{[\lambda\mu]}\right) + \tilde{R}^{\sigma}{}_{\sigma\lambda[\mu}g_{\nu]\rho} \\ & - g_{\lambda\nu}\left(2\tilde{R}^{(T)}_{[\rho\mu]} + \hat{R}^{(Q)}_{[\rho\mu]}\right) - g_{\rho\mu}\left(2\tilde{R}^{(T)}_{[\lambda\nu]} + \hat{R}^{(Q)}_{[\lambda\nu]}\right) - \tilde{R}^{\sigma}{}_{\sigma\rho[\mu}g_{\nu]\lambda}\Big] . \end{aligned} $$

- Note that the generalised Weyl tensor ${}^{(1)}\tilde{W}_{\lambda\rho\mu\nu}$ has the same symmetries as the Riemannian Weyl tensor ${}^{(1)}\tilde{W}_{\lambda\rho\mu\nu} = -{}^{(1)}\tilde{W}_{\rho\lambda\mu\nu} = -{}^{(1)}\tilde{W}_{\lambda\rho\nu\mu}\,, {}^{(1)}\tilde{W}_{\lambda[\rho\mu\nu]} = {}^{(1)}\tilde{W}^{\lambda}{}_{\mu\lambda\nu} = 0$.

- Symmetric components
  $\tilde{R}_{(\lambda\rho)\mu\nu} = \tilde{Z}_{\lambda\rho\mu\nu} = {}^{(1)}\tilde{Z}_{\lambda\rho\mu\nu} + {}^{(2)}\tilde{Z}_{\lambda\rho\mu\nu} + {}^{(3)}\tilde{Z}_{\lambda\rho\mu\nu} + {}^{(4)}\tilde{Z}_{\lambda\rho\mu\nu} + {}^{(5)}\tilde{Z}_{\lambda\rho\mu\nu}$:

$$ {}^{(1)}\tilde{Z}_{\lambda\rho\mu\nu} = \tilde{Z}_{\lambda\rho\mu\nu} - \sum_{i=2}^{5} {}^{(i)}\tilde{Z}_{\lambda\rho\mu\nu} \,, \tag{13} $$

$$ {}^{(2)}\tilde{Z}_{\lambda\rho\mu\nu} = \frac{1}{4}\left(\tilde{\not{R}}^{(Q)}_{\lambda[\rho\mu\nu]} + \tilde{\not{R}}^{(Q)}_{\rho[\lambda\mu\nu]}\right), \tag{14} $$

$$ {}^{(3)}\tilde{Z}_{\lambda\rho\mu\nu} = \frac{1}{6}\left(g_{\lambda\nu}\hat{R}^{(Q)}_{[\rho\mu]} + g_{\rho\nu}\hat{R}^{(Q)}_{[\lambda\mu]} - g_{\lambda\mu}\hat{R}^{(Q)}_{[\rho\nu]} - g_{\rho\mu}\hat{R}^{(Q)}_{[\lambda\nu]} + g_{\lambda\rho}\hat{R}^{(Q)}_{[\mu\nu]}\right), \tag{15} $$

$$ {}^{(4)}\tilde{Z}_{\lambda\rho\mu\nu} = \frac{1}{4} g_{\lambda\rho}\tilde{R}^{\sigma}{}_{\sigma\mu\nu} \,, \tag{16} $$

$$ {}^{(5)}\tilde{Z}_{\lambda\rho\mu\nu} = \frac{1}{8}\left(g_{\lambda\nu}\hat{\not{R}}^{(Q)}_{(\rho\mu)} + g_{\rho\nu}\hat{\not{R}}^{(Q)}_{(\lambda\mu)} - g_{\lambda\mu}\hat{\not{R}}^{(Q)}_{(\rho\nu)} - g_{\rho\mu}\hat{\not{R}}^{(Q)}_{(\lambda\nu)}\right). \tag{17} $$

# Curvature decomposition in Metric-Affine geometry

- Symmetric components
  $\tilde{R}_{(\lambda\rho)\mu\nu} = \tilde{Z}_{\lambda\rho\mu\nu} = {}^{(1)}\tilde{Z}_{\lambda\rho\mu\nu} + {}^{(2)}\tilde{Z}_{\lambda\rho\mu\nu} + {}^{(3)}\tilde{Z}_{\lambda\rho\mu\nu} + {}^{(4)}\tilde{Z}_{\lambda\rho\mu\nu} + {}^{(5)}\tilde{Z}_{\lambda\rho\mu\nu}$:

$$ {}^{(1)}\tilde{Z}_{\lambda\rho\mu\nu} = \tilde{Z}_{\lambda\rho\mu\nu} - \sum_{i=2}^{5} {}^{(i)}\tilde{Z}_{\lambda\rho\mu\nu} \,, \tag{13} $$

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$$ {}^{(3)}\tilde{Z}_{\lambda\rho\mu\nu} = \frac{1}{6}\left( g_{\lambda\nu}\hat{R}^{(Q)}_{[\rho\mu]} + g_{\rho\nu}\hat{R}^{(Q)}_{[\lambda\mu]} - g_{\lambda\mu}\hat{R}^{(Q)}_{[\rho\nu]} - g_{\rho\mu}\hat{R}^{(Q)}_{[\lambda\nu]} + g_{\lambda\rho}\hat{R}^{(Q)}_{[\mu\nu]} \right) , \tag{15} $$

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- The tensor ${}^{(2)}\tilde{Z}_{\lambda\rho\mu\nu}$ satisfies ${}^{(2)}\tilde{Z}_{(\lambda\rho\mu)\nu} = 0\,, {}^{(2)}\tilde{Z}^{\lambda}{}_{\lambda\mu\nu} = 0\,, {}^{(2)}\tilde{Z}^{\lambda}{}_{\mu\lambda\nu} = 0.$

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$\tilde{R}_{(\lambda\rho)\mu\nu} = \tilde{Z}_{\lambda\rho\mu\nu} = {}^{(1)}\tilde{Z}_{\lambda\rho\mu\nu} + {}^{(2)}\tilde{Z}_{\lambda\rho\mu\nu} + {}^{(3)}\tilde{Z}_{\lambda\rho\mu\nu} + {}^{(4)}\tilde{Z}_{\lambda\rho\mu\nu} + {}^{(5)}\tilde{Z}_{\lambda\rho\mu\nu}$:

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- Recall that if $Q_{\alpha\beta\mu} = 0$, then $\tilde{R}_{(\lambda\rho)\mu\nu} = \tilde{Z}_{\lambda\rho\mu\nu} = 0$.

# Curvature decomposition in Metric-Affine geometry

- Symmetric components
  $\tilde{R}_{(\lambda\rho)\mu\nu} = \tilde{Z}_{\lambda\rho\mu\nu} = {}^{(1)}\tilde{Z}_{\lambda\rho\mu\nu} + {}^{(2)}\tilde{Z}_{\lambda\rho\mu\nu} + {}^{(3)}\tilde{Z}_{\lambda\rho\mu\nu} + {}^{(4)}\tilde{Z}_{\lambda\rho\mu\nu} + {}^{(5)}\tilde{Z}_{\lambda\rho\mu\nu}$:

$$ {}^{(1)}\tilde{Z}_{\lambda\rho\mu\nu} = \tilde{Z}_{\lambda\rho\mu\nu} - \sum_{i=2}^{5} {}^{(i)}\tilde{Z}_{\lambda\rho\mu\nu} \,, \quad (13)$$

$$ {}^{(2)}\tilde{Z}_{\lambda\rho\mu\nu} = \frac{1}{4}\left(\not{\tilde{R}}^{(Q)}_{\lambda[\rho\mu\nu]} + \not{\tilde{R}}^{(Q)}_{\rho[\lambda\mu\nu]}\right) , \quad (14)$$

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- The tensor ${}^{(2)}\tilde{Z}_{\lambda\rho\mu\nu}$ satisfies ${}^{(2)}\tilde{Z}_{(\lambda\rho\mu)\nu} = 0\,,\, {}^{(2)}\tilde{Z}^{\lambda}{}_{\lambda\mu\nu} = 0\,,\, {}^{(2)}\tilde{Z}^{\lambda}{}_{\mu\lambda\nu} = 0$.
- Recall that if $Q_{\alpha\beta\mu} = 0$, then $\tilde{R}_{(\lambda\rho)\mu\nu} = \tilde{Z}_{\lambda\rho\mu\nu} = 0$.
- Special case that we will be interested: Weyl-Cartan geometry which is when $Q_{\alpha\beta\mu} = \frac{1}{4}g_{\mu\nu}Q_{\lambda\rho}{}^{\rho} + \not{Q}_{\lambda\mu\nu} = \frac{1}{4}g_{\mu\nu}Q_{\lambda\rho}{}^{\rho}$ (its traceless part is zero).

# Algebraic classification in Weyl-Cartan geometry

- As said before, Weyl-Cartan geometries are the ones where there is curvature, torsion and nonmetricity but $Q_{\alpha\beta\mu} = \frac{1}{4} g_{\mu\nu} Q_{\lambda\rho}{}^{\rho} + \not{Q}_{\lambda\mu\nu} = W_\alpha g_{\beta\mu}$, where $W_\mu = \frac{1}{4} Q_{\mu\nu}{}^{\nu}$ is called the Weyl vector.

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- In this particular geometry, only ${}^{(4)}\tilde{Z}_{\lambda\rho\mu\nu} = \frac{1}{4} g_{\lambda\rho} \tilde{R}^{\sigma}{}_{\sigma\mu\nu} = 4\nabla_{[\nu} W_{\mu]}$ is different from zero and all ${}^{(i)}\tilde{W}_{\lambda\rho\mu\nu}$ are also non-vanishing.

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- Then, we only have 7 building blocks:

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## Building blocks for Weyl-Cartan geometry

$$\left\{ {}^{(1)}\tilde{W}_{\lambda\rho\mu\nu}, \not{\tilde{R}}^{(T)}_{\lambda[\rho\mu\nu]}, \not{\tilde{R}}_{(\mu\nu)}, \tilde{R}^{(T)}_{[\mu\nu]}, \tilde{R}^{\lambda}{}_{\lambda\mu\nu}, \tilde{R}, *\tilde{R} \right\}.$$

$$\#10 + \#9 + \#9 + \#6 + \#6 + \#1 + \#1 = \#42\,\text{dof}$$

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## Building blocks for Weyl-Cartan geometry

$$\left\{ {}^{(1)}\tilde{W}_{\lambda\rho\mu\nu}, \not{\tilde{R}}^{(T)}_{\lambda[\rho\mu\nu]}, \not{\tilde{R}}_{(\mu\nu)}, \tilde{R}^{(T)}_{[\mu\nu]}, \tilde{R}^{\lambda}{}_{\lambda\mu\nu}, \tilde{R}, *\tilde{R} \right\}.$$

$$\#10 + \#9 + \#9 + \#6 + \#6 + \#1 + \#1 = \#42\,\mathrm{dof}$$

- The classification of $\left\{ \not{\tilde{R}}^{(T)}_{\lambda[\rho\mu\nu]}, \not{\tilde{R}}_{(\mu\nu)} \right\}$ are equivalent since they carry the same number of dof $\#9$.

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- In this particular geometry, only ${}^{(4)}\tilde{Z}_{\lambda\rho\mu\nu} = \frac{1}{4} g_{\lambda\rho} \tilde{R}^{\sigma}{}_{\sigma\mu\nu} = 4\nabla_{[\nu} W_{\mu]}$ is different from zero and all ${}^{(i)}\tilde{W}_{\lambda\rho\mu\nu}$ are also non-vanishing.
- Then, we only have 7 building blocks:

## Building blocks for Weyl-Cartan geometry

$$\left\{ {}^{(1)}\tilde{W}_{\lambda\rho\mu\nu}, \tilde{\not{R}}^{(T)}_{\lambda[\rho\mu\nu]}, \tilde{\not{R}}_{(\mu\nu)}, \tilde{R}^{(T)}_{[\mu\nu]}, \tilde{R}^{\lambda}{}_{\lambda\mu\nu}, \tilde{R}, *\tilde{R} \right\}.$$
$$\#10 + \#9 + \#9 + \#6 + \#6 + \#1 + \#1 = \#42\,\mathrm{dof}$$

- The classification of $\left\{ \tilde{\not{R}}^{(T)}_{\lambda[\rho\mu\nu]}, \tilde{\not{R}}_{(\mu\nu)} \right\}$ are equivalent since they carry the same number of dof $\#9$.
- Further, $\left\{ \tilde{R}^{(T)}_{[\mu\nu]}, \tilde{R}^{\lambda}{}_{\lambda\mu\nu} \right\}$ are equivalent since they also carry the same number of dof $\#6$.

# Algebraic types of ${}^{(1)}\tilde{W}_{\lambda\rho\mu\nu}$ ($\#10$ dof)

- We distribute its $10$ independent components in terms of the matrix quantity

$$ {}^{(1)}\tilde{W}^{+}_{\hat{0}a\hat{0}b} = {}^{(1)}\tilde{W}_{\hat{0}a\hat{0}b} + i *^{(1)} \tilde{W}_{\hat{0}a\hat{0}b} \,. \tag{18} $$

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$$ {}^{(1)}\tilde{W}^{+}_{\hat{0}a\hat{0}b} v^{b} = \lambda v_{a} \,. \tag{19} $$

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- Eigenvalue problem:

$$ {}^{(1)}\tilde{W}^{+}_{\hat{0}a\hat{0}b} v^{b} = \lambda v_{a} \,. \tag{19} $$

- Quadratic and cubic invariants:

$$ I = {}^{(1)}\tilde{W}^{+}_{abcd} {}^{(1)}\tilde{W}^{abcd} \,, \quad J = {}^{(1)}\tilde{W}^{+}_{ab}{}^{cd\,(1)}\tilde{W}_{cd}{}^{ef\,(1)}\tilde{W}_{ef}{}^{ab} \,. \tag{20} $$

# Algebraic types of ${}^{(1)}\tilde{W}_{\lambda\rho\mu\nu}$ ($\#10$ dof)

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- Characteristic polynomial:

$$ p(\lambda) = \lambda^{3} - \frac{I}{16}\lambda - \frac{J}{48} \,. \tag{21} $$

# Algebraic types of ${}^{(1)}\tilde{W}_{\lambda\rho\mu\nu}$ (#10 dof)

- Each special type satisfies a constraint associated with those null vectors $\{k_\mu, l_\mu, m_\mu, \bar{m}_\mu\}$ aligned with the tensor ${}^{(1)}\tilde{W}_{\lambda\rho\mu\nu}$ or principal null directions

$$k_{[\sigma}{}^{(1)}\tilde{W}_{\lambda]\rho\mu[\nu}k_{\omega]}k^\rho k^\mu = l_{[\sigma}{}^{(1)}\tilde{W}_{\lambda]\rho\mu[\nu}l_{\omega]}l^\rho l^\mu = 0\,, \tag{22}$$

$$m_{[\sigma}{}^{(1)}\tilde{W}_{\lambda]\rho\mu[\nu}m_{\omega]}m^\rho m^\mu = \bar{m}_{[\sigma}{}^{(1)}\tilde{W}_{\lambda]\rho\mu[\nu}\bar{m}_{\omega]}\bar{m}^\rho \bar{m}^\mu = 0\,. \tag{23}$$

where

$$k^\mu l_\mu = -m^\mu \bar{m}_\mu = 1\,, \tag{24}$$

$$k^\mu m_\mu = k^\mu \bar{m}_\mu = l^\mu m_\mu = l^\mu \bar{m}_\mu = 0\,, \tag{25}$$

$$k^\mu k_\mu = l^\mu l_\mu = m^\mu m_\mu = \bar{m}^\mu \bar{m}_\mu = 0\,, \tag{26}$$

and $g^{\mu\nu} = k^\mu l^\mu + k^\nu l^\mu + m^\mu \bar{m}^\nu + m^\nu \bar{m}^\mu$.

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$$k_{[\sigma}\,{}^{(1)}\tilde{W}_{\lambda]\rho\mu[\nu}k_{\omega]}k^\rho k^\mu = l_{[\sigma}\,{}^{(1)}\tilde{W}_{\lambda]\rho\mu[\nu}l_{\omega]}l^\rho l^\mu = 0\,, \tag{22}$$

$$m_{[\sigma}\,{}^{(1)}\tilde{W}_{\lambda]\rho\mu[\nu}m_{\omega]}m^\rho m^\mu = \bar{m}_{[\sigma}\,{}^{(1)}\tilde{W}_{\lambda]\rho\mu[\nu}\bar{m}_{\omega]}\bar{m}^\rho \bar{m}^\mu = 0\,. \tag{23}$$

where

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$$k^\mu k_\mu = l^\mu l_\mu = m^\mu m_\mu = \bar{m}^\mu \bar{m}_\mu = 0\,, \tag{26}$$

and $g^{\mu\nu} = k^\mu l^\mu + k^\nu l^\mu + m^\mu \bar{m}^\nu + m^\nu \bar{m}^\mu$.

- Constraints with the PNDs:

$$\begin{aligned}
l_{[\sigma}\,{}^{(1)}\tilde{W}_{\lambda]\rho\mu[\nu}l_{\omega]}l^\rho l^\mu = 0 &\iff \Sigma_0 = 0\,,\\
{}^{(1)}\tilde{W}_{\lambda\rho\mu[\nu}l_{\omega]}l^\rho l^\mu = 0 &\iff \Sigma_0 = \Sigma_1 = 0\,,\\
{}^{(1)}\tilde{W}_{\lambda\rho\mu[\nu}k_{\omega]}k^\rho k^\mu = {}^{(1)}\tilde{W}_{\lambda\rho\mu[\nu}l_{\omega]}l^\rho l^\mu = 0 &\iff \Sigma_0 = \Sigma_1 = \Sigma_3 = \Sigma_4 = 0\,,\\
{}^{(1)}\tilde{W}_{\lambda\rho\mu[\nu}l_{\omega]}l^\mu = 0 &\iff \Sigma_0 = \Sigma_1 = \Sigma_2 = 0\,,\\
{}^{(1)}\tilde{W}_{\lambda\rho\mu\nu}l^\mu = 0 &\iff \Sigma_0 = \Sigma_1 = \Sigma_2 = \Sigma_3 = 0\,.
\end{aligned} \tag{27}$$

where

$$\begin{aligned}
&\Sigma_0 = {}^{(1)}\tilde{W}_{\lambda\rho\nu\mu}l^\lambda m^\rho l^\mu m^\nu\,, \Sigma_1 = {}^{(1)}\tilde{W}_{\lambda\rho\nu\mu}l^\lambda k^\rho l^\mu m^\nu\,, \Sigma_2 = {}^{(1)}\tilde{W}_{\lambda\rho\nu\mu}l^\lambda m^\rho \bar{m}^\mu k^\nu\,,\\
&\Sigma_3 = {}^{(1)}\tilde{W}_{\lambda\rho\nu\mu}l^\lambda k^\rho \bar{m}^\mu k^\nu\,, \Sigma_4 = {}^{(1)}\tilde{W}_{\lambda\rho\nu\mu}k^\lambda \bar{m}^\rho k^\mu \bar{m}^\nu\,.
\end{aligned} \tag{28}$$

# Algebraic types of ${}^{(1)}\tilde{W}_{\lambda\rho\mu\nu}$ (#10 dof)

| Algebraic type | Description | Invariants | Constraints with the PNDs |
|---|---|---|---|
| $I$ | 3 eigenvectors and 3 eigenvalues | $I^3 \neq 12J^2$ | No further constraints |
| $II$ | 2 eigenvectors and 2 eigenvalues | $I^3 = 12J^2$ | ${}^{(1)}\tilde{W}_{\lambda\rho\mu[\nu}l_{\omega]}l^\rho l^\mu = 0$ |
| $D$ | 3 eigenvectors and 2 eigenvalues | $I^3 = 12J^2$ | ${}^{(1)}\tilde{W}_{\lambda\rho\mu[\nu}k_{\omega]}k^\rho k^\mu = {}^{(1)}\tilde{W}_{\lambda\rho\mu[\nu}l_{\omega]}l^\rho l^\mu = 0$ |
| $III$ | 1 eigenvectors and 1 eigenvalue | $I = J = 0$ | ${}^{(1)}\tilde{W}_{\lambda\rho\mu[\nu}l_{\omega]}l^\mu = 0$ |
| $N$ | 2 eigenvectors and 1 eigenvalue | $I = J = 0$ | ${}^{(1)}\tilde{W}_{\lambda\rho\mu\nu}l^\mu = 0$ |
| $O$ | $[-]$ | $I = J = 0$ | ${}^{(1)}\tilde{W}_{\lambda\rho\mu\nu} = 0$ |

Table: Algebraic types for the tensor ${}^{(1)}\tilde{W}_{\lambda\rho\mu\nu}$.

# Algebraic types of $\tilde{R}^{(T)}_{[\mu\nu]}$ and $\tilde{R}^{\lambda}{}_{\lambda\mu\nu}$ (#9 dof each)

- Components collected in the symmetric spinors:

$$\Xi_{AB} = \frac{1}{2}\varepsilon^{\dot{C}\dot{D}}\tau^{\mu}{}_{A\dot{C}}\tau^{\nu}{}_{B\dot{D}}\tilde{R}^{(T)}_{[\mu\nu]}\,, \tag{29}$$

$$\Upsilon_{AB} = \frac{1}{2}\varepsilon^{\dot{C}\dot{D}}\tau^{\mu}{}_{A\dot{C}}\tau^{\nu}{}_{B\dot{D}}\tilde{R}^{\lambda}{}_{\lambda\mu\nu}\,. \tag{30}$$

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- Eigenvalue problem:

$$\Xi^{A}{}_{B}\xi^{B} = \lambda\xi^{A}\,, \quad (31)$$

$$\Upsilon^{A}{}_{B}\zeta^{B} = \sigma\zeta^{A}\,. \quad (32)$$

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- Quadratic invariants:

$$X = \Xi_{AB}\Xi^{AB}\,, \quad Y = \Upsilon_{AB}\Upsilon^{AB}\,. \tag{33}$$

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$$\Xi_{AB} = \frac{1}{2}\varepsilon^{\dot{C}\dot{D}}\tau^{\mu}{}_{A\dot{C}}\tau^{\nu}{}_{B\dot{D}}\tilde{R}^{(T)}_{[\mu\nu]}\,, \tag{29}$$

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- Eigenvalue problem:

$$\Xi^{A}{}_{B}\xi^{B} = \lambda\xi^{A}\,, \tag{31}$$

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- Quadratic invariants:

$$X = \Xi_{AB}\Xi^{AB}\,, \quad Y = \Upsilon_{AB}\Upsilon^{AB}\,. \tag{33}$$

- Set of eigenvalues:

$$\lambda = \pm\sqrt{-\frac{X}{2}}\,, \quad \sigma = \pm\sqrt{-\frac{Y}{2}}\,. \tag{34}$$

# Algebraic types of $\tilde{R}^{(T)}_{[\mu\nu]}$ and $\tilde{R}^{\lambda}{}_{\lambda\mu\nu}$ (#$9$ dof each)

- Expressions of $\tilde{R}^{(T)}_{[\mu\nu]}$ and $\tilde{R}^{\lambda}{}_{\lambda\mu\nu}$ in terms of null basis:

$$\begin{aligned}\tilde{R}^{(T)}_{[\mu\nu]} &= 2\big[\Omega_2 k_{[\mu}m_{\nu]} + \bar{\Omega}_2 k_{[\mu}\bar{m}_{\nu]} - \Omega_0 l_{[\mu}\bar{m}_{\nu]} - \bar{\Omega}_0 l_{[\mu}m_{\nu]} \\ &\quad - \left(\Omega_1 + \bar{\Omega}_1\right) k_{[\mu}l_{\nu]} + \left(\Omega_1 - \bar{\Omega}_1\right) m_{[\mu}\bar{m}_{\nu]}\big]\,, \qquad (35)\\ \tilde{R}^{\lambda}{}_{\lambda\mu\nu} &= 2\big[\Pi_2 k_{[\mu}m_{\nu]} + \bar{\Pi}_2 k_{[\mu}\bar{m}_{\nu]} - \Pi_0 l_{[\mu}\bar{m}_{\nu]} - \bar{\Pi}_0 l_{[\mu}m_{\nu]} \\ &\quad - \left(\Pi_1 + \bar{\Pi}_1\right) k_{[\mu}l_{\nu]} + \left(\Pi_1 - \bar{\Pi}_1\right) m_{[\mu}\bar{m}_{\nu]}\big]\,, \qquad (36)\end{aligned}$$

where

$$\Omega_0 = k^{[\mu}m^{\nu]}\tilde{R}^{(T)}_{[\mu\nu]}\,, \quad \Omega_1 = \frac{1}{2}\left(k^{[\mu}l^{\nu]} - m^{[\mu}\bar{m}^{\nu]}\right)\tilde{R}^{(T)}_{[\mu\nu]}\,, \quad \Omega_2 = -\,l^{[\mu}\bar{m}^{\nu]}\tilde{R}^{(T)}_{[\mu\nu]}\,,$$

$$\Pi_0 = k^{[\mu}m^{\nu]}\tilde{R}^{\lambda}{}_{\lambda\mu\nu}\,, \quad \Pi_1 = \frac{1}{2}\left(k^{[\mu}l^{\nu]} - m^{[\mu}\bar{m}^{\nu]}\right)\tilde{R}^{\lambda}{}_{\lambda\mu\nu}\,, \quad \Pi_2 = -\,l^{[\mu}\bar{m}^{\nu]}\tilde{R}^{\lambda}{}_{\lambda\mu\nu}\,.$$

# Algebraic types of $\tilde{R}^{(T)}_{[\mu\nu]}$ and $\tilde{R}^{\lambda}{}_{\lambda\mu\nu}$ (#9 dof each)

- PNDs:

$$\left(\tilde{R}^{(T)}_{[\mu\nu]}k_\lambda - \tilde{R}^{(T)}_{[\mu\lambda]}k_\nu\right)k^\mu = \left(\tilde{R}^{(T)}_{[\mu\nu]}l_\lambda - \tilde{R}^{(T)}_{[\mu\lambda]}l_\nu\right)l^\mu = 0\,, \tag{37}$$

$$\left(\tilde{R}^{\sigma}{}_{\sigma\mu\nu}k_\lambda - \tilde{R}^{\sigma}{}_{\sigma\mu\lambda}k_\nu\right)k^\mu = \left(\tilde{R}^{\sigma}{}_{\sigma\mu\nu}l_\lambda - \tilde{R}^{\sigma}{}_{\sigma\mu\lambda}l_\nu\right)l^\mu = 0\,. \tag{38}$$

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- Constraints with the PNDs:

$$\big(\tilde{R}^{(T)}_{[\mu\nu]}l_\lambda - \tilde{R}^{(T)}_{[\mu\lambda]}l_\nu\big)l^\mu = 0 \iff \Omega_2 = 0\,, \tag{39}$$

$$\big(\tilde{R}^{\sigma}{}_{\sigma\mu\nu}l_\lambda - \tilde{R}^{\sigma}{}_{\sigma\mu\lambda}l_\nu\big)l^\mu = 0 \iff \Pi_2 = 0\,, \tag{40}$$

$$\big(\tilde{R}^{(T)}_{[\mu\nu]}k_\lambda - \tilde{R}^{(T)}_{[\mu\lambda]}k_\nu\big)k^\mu = 0 \iff \Omega_0 = 0\,, \tag{41}$$

$$\big(\tilde{R}^{\sigma}{}_{\sigma\mu\nu}k_\lambda - \tilde{R}^{\sigma}{}_{\sigma\mu\lambda}k_\nu\big)k^\mu = 0 \iff \Pi_0 = 0\,, \tag{42}$$

$$\tilde{R}^{(T)}_{[\mu\nu]}l^\mu = 0 \iff \Omega_1 = \Omega_2 = 0\,, \tag{43}$$

$$\tilde{R}^{\sigma}{}_{\sigma\mu\nu}l^\mu = 0 \iff \Pi_1 = \Pi_2 = 0\,. \tag{44}$$

# Algebraic types of $\tilde{R}^{(T)}_{[\mu\nu]}$ and $\tilde{R}^{\lambda}{}_{\lambda\mu\nu}$ (#9 dof each)

| Algebraic type | Segre characteristic | Invariants | Constraints with the PNDs |
|---|---|---|---|
| $I$ | $[1\,1]$ | $X \neq 0$ | No further constraints |
| $N$ | $[2]$ | $X = 0$ | $\tilde{R}^{(T)}_{[\mu\nu]} l^{\mu} = 0$ |
| $O$ | $[-]$ | $X = 0$ | $\tilde{R}^{(T)}_{[\mu\nu]} = 0$ |

Table: Algebraic types for the tensor $\tilde{R}^{(T)}_{[\mu\nu]}$.

| Algebraic type | Segre characteristic | Invariants | Constraints with the PNDs |
|---|---|---|---|
| $I$ | $[1\,1]$ | $Y \neq 0$ | No further constraints |
| $N$ | $[2]$ | $Y = 0$ | $\tilde{R}^{\lambda}{}_{\lambda\mu\nu} l^{\mu} = 0$ |
| $O$ | $[-]$ | $Y = 0$ | $\tilde{R}^{\lambda}{}_{\lambda\mu\nu} = 0$ |

Table: Algebraic types for the tensor $\tilde{R}^{\lambda}{}_{\lambda\mu\nu}$.

# Algebraic types of $\tilde{\not{R}}^{(T)}_{\lambda[\rho\mu\nu]}$ and $\tilde{\not{R}}_{(\mu\nu)}$ ($\#6$ dof each) (see our paper)

| Description | Invariants |
|---|---|
| 1 timelike eigenvector and 3 spacelike eigenvectors | $U_1^*, V_1^* > 0$ |
| 2 complex conjugate eigenvectors and 2 spacelike eigenvectors | $U_1^* < 0$ |
| 2 complex conjugate eigenvectors and 2 spacelike eigenvectors | $U_1^* = 0\,,\ V_1^* < 0\,,\ W_1^* > 0$ |
| 1 null eigenvector and 2 spacelike eigenvectors | $U_1^* = 0\,,\ V_1^* > 0\,,\ W_1^* > 0$ |
| 1 timelike eigenvector and 3 spacelike eigenvectors | $U_1^* = 0\,,\ V_1^* > 0\,,\ W_1^* > 0$ |
| 2 null eigenvectors and 2 spacelike eigenvectors | $U_1^* = 0\,,\ V_1^* > 0\,,\ W_1^* > 0$ |
| 1 null eigenvector and 1 spacelike eigenvector | $U_1^* = W_1^* = 0\,,\ V_1^* > 0$ |
| 1 null eigenvector and 2 spacelike eigenvectors | $U_1^* = W_1^* = 0\,,\ V_1^* > 0$ |
| 2 null eigenvectors and 2 spacelike eigenvectors | $U_1^* = W_1^* = 0\,,\ V_1^* > 0$ |
| 1 timelike eigenvector and 3 spacelike eigenvectors | $U_1^* = W_1^* = 0\,,\ V_1^* > 0$ |
| 1 null eigenvector and 2 spacelike eigenvectors | $U_1^* = V_1^* = 0\,,\ W_1^* > 0$ |
| 2 null eigenvectors and 2 spacelike eigenvectors | $U_1^* = V_1^* = 0\,,\ W_1^* > 0$ |
| 1 null eigenvector and 1 spacelike eigenvector | $U_1^* = V_1^* = W_1^* = 0$ |
| 1 null eigenvector and 2 spacelike eigenvectors | $U_1^* = V_1^* = W_1^* = 0$ |
| 2 null eigenvectors and 2 spacelike eigenvectors | $U_1^* = V_1^* = W_1^* = 0$ |

Table: Algebraic types for the tensor $\tilde{\not{R}}^{(T)}_{\lambda[\rho\mu\nu]}$.

- MAG model with dynamical torsion in $4$ dimensions[1]:

$$\mathcal{L} = \frac{1}{64\pi}\Big(-4R - 6d_1\tilde{R}_{\lambda[\rho\mu\nu]}\tilde{R}^{\lambda[\rho\mu\nu]} - 9d_1\tilde{R}_{\lambda[\rho\mu\nu]}\tilde{R}^{\mu[\lambda\nu\rho]} + 8\,d_1\tilde{R}_{[\mu\nu]}\tilde{R}^{[\mu\nu]}\Big)\,.$$

[1] S. Bahamonde, J. Chevrier and J. G. Valcarcel, JCAP **02**, 018 (2023).
[2] S. Bahamonde and J. G. Valcarcel, arXiv: 2305.05501 [gr-qc].

# Application to stationary and axisymmetric space-times

- MAG model with dynamical torsion in $4$ dimensions[1]:

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- In terms of building blocks:

$$\mathcal{L} = \frac{1}{64\pi}\Big(-4R - 6d_1\,\tilde{\not{R}}^{(T)}_{\lambda[\rho\mu\nu]}\,\tilde{\not{R}}^{(T)\,\lambda[\rho\mu\nu]} - 9d_1\,\tilde{\not{R}}^{(T)}_{\lambda[\rho\mu\nu]}\,\tilde{\not{R}}^{(T)\,\mu[\lambda\nu\rho]} + 2d_1\tilde{R}^{(T)}_{[\mu\nu]}\tilde{R}^{(T)\,[\mu\nu]} - \frac{d_1}{8} * \tilde{R}^2\Big)\,. \tag{45}$$

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- We showed that the field strength tensors of torsion cannot be doubly aligned with the principal null directions of the Riemannian Weyl tensor in generic stationary and axisymmetric space-times[2]:

$$W_{\lambda\rho\mu[\nu}k_{\omega]}k^{\rho}k^{\mu} = W_{\lambda\rho\mu[\nu}l_{\omega]}l^{\rho}l^{\mu} = 0\,, \tag{46}$$

$$\big(\tilde{R}^{(T)}_{[\mu\nu]}k_{\lambda} - \tilde{R}^{(T)}_{[\mu\lambda]}k_{\nu}\big)k^{\mu} = \big(\tilde{R}^{(T)}_{[\mu\nu]}l_{\lambda} - \tilde{R}^{(T)}_{[\mu\lambda]}l_{\nu}\big)l^{\mu} = 0\,. \tag{47}$$

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# Conclusions

- The algebraic classification of the curvature tensor has been largely analysed in the framework of Riemannian geometry and, particularly, in GR, while the corresponding studies and applications in MAG remain unexplored.

[3] S. Bahamonde and J. G. Valcarcel, arXiv: 2305.05501 [gr-qc].

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- The algebraic classification of the curvature tensor has been largely analysed in the framework of Riemannian geometry and, particularly, in GR, while the corresponding studies and applications in MAG remain unexplored.
- We performed an irreducible decomposition and algebraic classification of the curvature tensor are important analyses to perform in MAG.

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- We performed an irreducible decomposition and algebraic classification of the curvature tensor are important analyses to perform in MAG.
- From a general irreducible decomposition depending on $11$ building blocks, we derive the algebraic classification of the curvature tensor in Weyl-Cartan geometry[3].

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- We performed an irreducible decomposition and algebraic classification of the curvature tensor are important analyses to perform in MAG.
- From a general irreducible decomposition depending on $11$ building blocks, we derive the algebraic classification of the curvature tensor in Weyl-Cartan geometry[3].
- As an application, we demonstrate that for a MAG model based on scalar-flat geometries the field strength tensors of torsion cannot be doubly aligned with the principal null directions of the Riemannian Weyl tensor in generic stationary and axisymmetric space-times.

[3] S. Bahamonde and J. G. Valcarcel, arXiv: 2305.05501 [gr-qc].