New Exact Spherically Symmetric Solutions in $f(R,\phi,X)$ Gravity by Noether's Symmetry Approach

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Outline

1. Generalised $f(R, \phi, X)$ gravity
2. Noether symmetry and solutions
   - Symmetry reduce Lagrangian
   - Noether symmetry approach
   - Spherically symmetric solutions in $f(R, \phi, X)$
3. Conclusions

Generalised $f(R,\phi,X)$ gravity

- Let us consider the following action

Action considered

$$S=\int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}f(R,\phi,X)+L_{\rm m}\right]. \quad (1)$$

- Here, $\kappa^2=8\pi G$, $L_{\rm m}$ is any matter Lagrangian and f is a function which depends on the scalar curvature R, a scalar field ϕ and a kinetic term being equal to

$$X=-\frac{\epsilon}{2}\,\partial^\mu\phi\partial_\mu\phi\,, \quad (2)$$

where ϵ is a parameter ($\epsilon=-1$: phantom scalar field and $\epsilon=1$: canonical scalar field)

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Generalised $f(R,\phi,X)$ gravity

- Some interesting specific cases that can be constructed from this theory:
  - If $f(R,\phi,X)=f(R)$, one recovers $f(R)$ gravity.
  - If $f(R,\phi,X)=\alpha(R)+\gamma(X,\phi)$, one recovers theories minimally coupled with the scalar field.
  - If $f(R,\phi,X)=f(\phi,X)R$, one recovers Brans-Dicke type gravity theories and theories non-minimally couple between the scalar curvature and ϕ or X.
- This theory contains a large number of theories: some of them very interesting for cosmology and also for astrophysics objects.

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Generalised $f(R, \phi, X)$ gravity

- Variations of the action with respect to the metric yields

$$f_R G_{\mu\nu} = \frac{1}{2}(f - R f_R)\, g_{\mu\nu} + \nabla_\nu \nabla_\mu f_R - g_{\mu\nu} \nabla_\alpha \nabla^\alpha f_R + \frac{\epsilon}{2} f_X \,(\nabla_\mu \phi)(\nabla_\nu \phi)\,, \quad (3)$$

whereas variations with respect to the scalar field ϕ gives

$$\nabla_\mu \left(f_X \nabla^\mu \phi\right) + \epsilon f_\phi = 0\,. \quad (4)$$

- Here, $L_{\rm m} = 0$ (vacuum case) and $f_R = \partial f / \partial R$, $f_X = \partial f / \partial X$ and $f_\phi = \partial f / \partial \phi$

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Spherical symmetry in Generalised $f(R, \phi, X)$ gravity

- Let us now consider that the space-time is spherically symmetric such as the metric is:

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + M(r)(d\theta^2 + \sin\theta^2 d\varphi^2), \qquad (5)$$

where $A(r), B(r)$ and $M(r)$ are functions of the radial coordinate r.

- The scalar curvature becomes

$$R = -\frac{1}{B}\Big[\frac{A''}{A} + \frac{2M''}{M} - \frac{A'B'}{2AB} + \frac{M'A'}{MA} - \frac{M'B'}{MB} - \frac{A'^2}{2A^2} - \frac{M'^2}{2M^2} - \frac{2B}{M}\Big]. \qquad (6)$$

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The field equations for spherically symmetric space-times become

$$f_R\left(2\frac{A''}{A}-\frac{A'B'}{AB}-\frac{A'^2}{A^2}+2\frac{M'A'}{MA}\right)=-2Bf+4\left[f_R''+f_R'\left(\frac{M'}{M}-\frac{B'}{2B}\right)\right],$$

$$f_R\left(2\frac{A''}{A}+4\frac{M''}{M}-\frac{A'B'}{AB}-2\frac{M'B'}{MB}-\frac{A'^2}{A^2}-2\frac{M'^2}{M^2}\right)=-2Bf$$
$$+2f_R'\left(\frac{A'}{A}+2\frac{M'}{M}\right)-2\epsilon f_X\phi'^2,$$

$$f_R\left(2\frac{M''}{M}-\frac{B'M'}{BM}+\frac{M'A'}{MA}-\frac{4B}{M}\right)=-2Bf+4\left[f_R''+\frac{1}{2}f_R'\left(\frac{A'}{A}-\frac{B'}{B}+\frac{M'}{M}\right)\right].$$

where primes denote differentiation with respect to r.
The modified Klein-Gordon equation yields

$$f_X\left[\phi''+\frac{1}{2}\phi'\left(\frac{A'}{A}-\frac{B'}{B}+2\frac{M'}{M}\right)\right]+f_X'\phi'+\epsilon Bf_\phi=0. \tag{7}$$

Spherical symmetry in Generalised $f(R, \phi, X)$ gravity

- Schwarzschild solution is the unique spherically symmetric vacuum solution in GR, but we will see that this no longer holds in $f(R, \phi, X)$ theory of gravity.
- Vacuum solutions do not necessarily imply a null curvature $R = 0$ or $R = const.$, which lead to maximally symmetric solutions, to the contrary in GR.
- The field equations are difficult to treat and it is not so easy to find new spherically symmetric solutions.
- Nowadays, it is vital to explore spherically symmetric solutions in modified gravity to then try to analyse them with observations in order to see any new effects different to GR.

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Spherical symmetry in Generalised $f(R, \phi, X)$ gravity

Important point 1

Many papers in modified gravity have found spherically symmetric solutions considering the case $R = \text{constant}$. Although this case has been widely studied in the literature, this case is more or less trivial: All the higher order terms are zero and therefore, the underlying theory becomes similar to GR+Λ.

Important point 2

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Spherical symmetry in Generalised $f(R, \phi, X)$ gravity

Our approach

In our approach, WE **WILL NOT ASSUME** $R = \text{constant}$, otherwise the theory becomes the same as GR. In this case, finding solutions become a very difficult task.

How can we find solutions?

We adopt the classical Noether approach in order to find the Noether symmetry in $f(R, \phi, X)$ gravity. From the classical Noether theorem, it is shown that the Noether symmetry in $f(R, \phi, X)$ gravity leads to a kind of the first integral of motion, which are able to be solved. We derive exact solutions for the field equations by using the conservation relation coming from the Noether symmetry acquired

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Symmetry reduce Lagrangian for $f(R,\phi,X)$

- One can rewrite the action into its canonical form in such a way that we can reduce the number of degrees of freedom:

$$S_{f(R,\phi,X)} = \int dr \mathcal{L}(A, A', B, B', M, M', R, R', \phi, \phi') \,. \quad (8)$$

- For simplicity let us express the scalar curvature as follows

$$\bar{R} = R^* - \frac{A''}{AB} - \frac{2M''}{BM} \,, \quad (9)$$

where $R^* = \frac{A'B'}{2AB^2} - \frac{A'M'}{ABM} + \frac{A'^2}{2A^2B} + \frac{B'M'}{B^2M} + \frac{M'^2}{2BM^2} + \frac{2}{M}$ contains only first derivatives terms.

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- The $f(R,\phi,X)$ action in spherically symmetric space-time becomes

$$S_{f(R,\phi,X)} = \int dr \Big\{ f(R,\phi,X) - \lambda_1 \left(R - \bar{R}\right) - \lambda_2 \left(X - \bar{X}\right) \Big\} M\sqrt{AB}\,,$$

  where $\bar{X} = -\frac{\epsilon}{2B}\,\phi'^2$ and λ_1 and λ_2 are the Lagrangian multipliers.
- By varying with respect to R and X: $\lambda_1 = f_R$ and $\lambda_2 = f_X$.

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- Then, the canonical action can be rewritten as

$$\begin{aligned} S_{f(R,\phi,X)} \quad = \quad & \int dr \Big\{ M\sqrt{AB}\Big[f(R,\phi,X) - f_R(R - R^*) \Big] \\ & +2M'\Big(\sqrt{\frac{A}{B}} f_R\Big)' + A'\Big(\frac{M f_R}{\sqrt{AB}}\Big)' \\ & -M\sqrt{AB} f_X \left(X + \frac{\epsilon}{2B}\, \phi'^2 \right) \Big\}, \quad (10) \end{aligned}$$

where boundary terms were ignored.

Symmetry reduce Lagrangian for $f(R,\phi,X)$

- Then, the symmetry reduced Lagrangian becomes

$$\begin{aligned}\mathcal{L}_f &= \sqrt{AB}\left[M(f - Xf_X) + (2 - MR)f_R\right] + \frac{f_R M' A'}{\sqrt{AB}} \\ &+\frac{1}{2} f_R \sqrt{\frac{A}{B}}\frac{M'^2}{M} + \frac{MA' f_R'}{\sqrt{AB}} + 2\sqrt{\frac{A}{B}} f_R' M' \\ &-\frac{\epsilon}{2} f_X M \sqrt{\frac{A}{B}}\phi'^2 \,. \end{aligned} \tag{11}$$

Remark: $f_R' = f_{RR}R' + f_{R\phi}\phi' + f_{RX}X'$.

- As we saw before, in the second field equation, there is no derivatives in $B(r)$. Hence, it can be explicitly solved in terms of B.

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Symmetry reduce Lagrangian for $f(R, \phi, X)$

- The metric variable B does not contribute to the dynamics due to the symmetry reduced Lagrangian approach, but the equation of motion for B has to be considered as a further constraint equation.
- By considering this constraint, one can reduce the point-like Lagrangian to:

$$\mathcal{L} = [M(f - Xf_X) + (2 - MR)f_R]\left[f_R\left(M'A' + \frac{AM'^2}{2M}\right) + f_R'(MA' + 2AM') - \frac{\epsilon}{2}f_X MA\phi'^2\right]. \quad (12)$$

- Therefore, we can consider $\mathcal{L}$ as the new Lagrangian with five degrees of freedom (A, M, R, ϕ and X).

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- The energy functional $E_{\mathcal{L}}$ or the Hamiltonian of the Lagrangian $\mathcal{L}$ is defined by

$$E_{\mathcal{L}} = q'^i \frac{\partial \mathcal{L}}{\partial q'^i} - \mathcal{L}. \tag{13}$$

- In our case, the Hamiltonian becomes

$$E_{\mathcal{L}} = AB \left[M(f - X f_X) + (2 - MR) f_R \right]^2 , \tag{14}$$

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Noether symmetry approach

- Let us introduce a generator of the form

$$\mathbf{Y} = \xi \frac{\partial}{\partial r} + \eta^i \frac{\partial}{\partial q^i}, \tag{15}$$

where q^i are the generalized coordinates in the d-dimensional configuration space $\mathcal{Q} \equiv \{q^i, i = 1, \ldots, d\}$ of the Lagrangian, whose tangent space is $\mathcal{TQ} \equiv \{q^i, q'^i\}$.

- The components ξ and η^i of the Noether symmetry generator $\mathbf{Y}$ are functions of r and q^i.

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Noether symmetry approach

- The existence of a Noether symmetry implies the existence of a vector field $\mathbf{Y}$ if the Lagrangian $\mathcal{L}(r, q^i, q'^i)$ satisfies

Noether symmetry condition

$$\mathbf{Y}^{[1]}\mathcal{L} + \mathcal{L}(D_r\xi) = D_r K\,, \tag{16}$$

where $\mathbf{Y}^{[1]}$ is the first prolongation of the generator (15) in such a form

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and $K(r, q^i)$ is a gauge function, D_r is the total derivative operator with respect to r, $D_r = \partial/\partial r + q'^i \partial/\partial q^i$, and η'^i is defined as $\eta'^i = D_r \eta^i - q'^i D_r \xi$.

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Noether symmetry approach

- The significance of Noether symmetry comes from the following first integral of motion that if $\mathbf{Y}$ is the Noether symmetry generator corresponding to the Lagrangian $\mathcal{L}(r, q^i, q'^i)$, then the Hamiltonian or a conserved quantity associated with the generator $\mathbf{Y}$ is

$$I = -\xi E_{\mathcal{L}} + \eta^i \frac{\partial \mathcal{L}}{\partial q'^i} - K, \tag{18}$$

where I is a constant of motion *or* Noether constant.

Noether symmetry approach for $f(R, \phi, X)$

- By using the reduced Lagrangian for $f(R, \phi, X)$ in spherically symmetric space-time into the Noether's condition one gets a system of 26 partial differential equations.
- To find the symmetries and the integral of motion, one needs to specify the model of f, and be brave to solve the system.

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\begin{aligned}
&\xi_{,A}=0,\quad \xi_{,M}=0,\quad \xi_{,R}=0,\quad \xi_{,\phi}=0,\quad \xi_{,X}=0,\quad K_{,r}=0,\\
&f_R\eta^2_{,r}+M\left(f_{RR}\eta^3_{,r}+f_{R\phi}\eta^4_{,r}+f_{RX}\eta^5_{,r}\right)-\frac{1}{F}K_{,A}=0,\quad f_{RX}\left(M\eta^1_{,r}+A\eta^2_{,r}\right)-\frac{1}{F}K_{,X}=0,\\
&f_R\left(\eta^1_{,r}+\frac{A}{M}\eta^2_{,r}\right)+2A\left(f_{RR}\eta^3_{,r}+f_{R\phi}\eta^4_{,r}+f_{RX}\eta^5_{,r}\right)-\frac{1}{F}K_{,M}=0,\\
&f_{RR}\left(M\eta^1_{,r}+A\eta^2_{,r}\right)-\frac{1}{F}K_{,R}=0,\quad f_{RR}\left(M\eta^1_{,r}+A\eta^2_{,r}\right)-\epsilon MAf_X\eta^4_{,r}-\frac{1}{F}K_{,\phi}=0,\\
&f_{RR}\left(M\eta^1_{,R}+2A\eta^2_{,R}\right)=0,\quad f_{RR}\left(M\eta^1_{,\phi}+2A\eta^2_{,\phi}\right)+f_{R\phi}\left(M\eta^1_{,R}+2A\eta^2_{,R}\right)=0,\\
&f_{RR}\left(M\eta^1_{,X}+2A\eta^2_{,X}\right)+f_{RX}\left(M\eta^1_{,R}+2A\eta^2_{,R}\right)=0,\quad f_{RX}\left(M\eta^1_{,X}+2A\eta^2_{,X}\right)=0,\\
&f_{R\phi}\left(M\eta^1_{,X}+2A\eta^2_{,X}\right)+f_{RX}\left(M\eta^1_{,\phi}+2A\eta^2_{,\phi}\right)-\epsilon MAf_X\eta^4_{,X}=0,\quad f_R\eta^2_{,A}+M\left(f_{RR}\eta^3_{,A}+f_{R\phi}\eta^4_{,A}+f_{RX}\eta^5_{,A}\right)=0,\\
&\quad f_R\left(\eta^1_{,A}+\frac{A}{M}\eta^2_{,A}+\eta^2_{,M}-\xi_{,r}\right)+M\left(f_{RR}\eta^3_{,M}+f_{R\phi}\eta^4_{,M}+f_{RX}\eta^5_{,M}\right)+2A\left(f_{RR}\eta^3_{,A}+f_{R\phi}\eta^4_{,A}+f_{RX}\eta^5_{,A}\right)\\
&\qquad+\frac{1}{F}\left[f_RF_{,M}\eta^2+\left(Ff_R\right)_{,R}\eta^3+\left(Ff_R\right)_{,\phi}\eta^4+\left(Ff_R\right)_{,X}\eta^5\right]=0,\\
&\qquad\quad\frac{f_R}{M}\eta^2_{,R}+f_{RR}\left(\eta^1_{,A}+\frac{2A}{M}\eta^2_{,A}+\eta^3_{,R}-\xi_{,r}\right)+f_{R\phi}\eta^4_{,R}+f_{RX}\eta^5_{,R}\\
&\qquad\qquad+\frac{1}{F}\left[f_{RR}(FM)_{,M}\frac{\eta^2}{M}+\left(Ff_{RR}\right)_{,R}\eta^3+\left(Ff_{RR}\right)_{,\phi}\eta^4+\left(Ff_{RR}\right)_{,X}\eta^5\right]=0,\\
&\qquad\quad\frac{f_R}{M}\eta^2_{,\phi}+f_{R\phi}\left(\eta^1_{,A}+\frac{2A}{M}\eta^2_{,A}+\eta^4_{,\phi}-\xi_{,r}\right)+f_{RR}\eta^3_{,\phi}+f_{RX}\eta^5_{,\phi}\\
&\qquad\qquad+\frac{1}{F}\left[f_{R\phi}(FM)_{,M}\frac{\eta^2}{M}+\left(Ff_{R\phi}\right)_{,R}\eta^3+\left(Ff_{R\phi}\right)_{,\phi}\eta^4+\left(Ff_{R\phi}\right)_{,X}\eta^5\right]=0,\\
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&\qquad\quad+\frac{1}{F}\left[\left(Ff_R\right)_{,R}\eta^3+\left(Ff_R\right)_{,\phi}\eta^4+\left(Ff_R\right)_{,X}\eta^5\right]=0,\\
&\qquad\frac{f_R}{2A}\left(\eta^1_{,R}+\frac{A}{M}\eta^2_{,R}\right)+f_{RR}\left[\frac{\eta^1}{A}+\frac{F_{,M}}{F}\eta^2+\frac{M}{2A}\eta^1_{,M}+\eta^2_{,M}+\eta^3_{,R}-\xi_{,r}\right]+f_{R\phi}\eta^4_{,R}+f_{RX}\eta^5_{,R}\\
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&\qquad\qquad\quad+\frac{1}{F}\left[\left(Ff_X\right)_{,R}\eta^3+\left(Ff_R\right)_{,\phi}\eta^4+\left(Ff_X\right)_{,X}\eta^5\right]=0,
\end{aligned}
\tag{1}
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Noether symmetry approach for $f(R,\phi,X)$: cases

We will split the study in 4 different types of $f(R,\phi,X)$:

1. $f(R,\phi,X) = f_0 R^n$, **Power-law** $f(R)$
2. $f(R,\phi,X) = f_0 R + f_1 X^q - V(\phi)$, **Minimally coupled theory.**
3. $f(R,\phi,X) = f_0 \phi^m R^n + f_1 X^q - V(\phi)$, **non-minimally coupled theory between the scalar field and** R**.**
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- Solving the Noether's conditions one gets

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\begin{aligned}
A &= R^{\frac{(n-1)(2n-1)}{n-2}} \left[A_0 + \frac{I_3}{f_0^2 n(2-n)} \int \frac{R^{\frac{(n-1)(4n-5)}{2-n}} dr}{M[2n+(1-n)MR]} \right], \\
B &= \frac{nM}{[2n+(1-n)MR]} \left[(n-1)\frac{R'}{R} \left(\frac{A'}{A} + \frac{2M'}{M} \right) \right. \\
&\quad \left. + \frac{M'}{M} \left(\frac{A'}{A} + \frac{M'}{2M} \right) \right].
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- When $n = 5/4, p = -2, R_0 = -5$ which gives $R(r) = -\frac{5}{r^2}$, one gets:

Analytical solution 1

$$A(r) = \frac{1}{\sqrt{5}}\left(k_1 r + k_2\right), \qquad B(r) = \frac{1}{2\left(1 + \frac{k_2}{k_1 r}\right)}, \tag{19}$$

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*S. Capozziello, A. Stabile and A. Troisi, Class. Quant. Grav. **24**, 2153 (2007).

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Case I: $f(R,\phi,X) = f_0 R^n$

- For $M(r) = r^2, R = R_0 r^p$ and $p = (n-2)/(4n^2-10n+7)$

Analytical solution 2

$$A(r) = r^{\frac{(n-1)(2n-1)}{4n^2-10n+7}}\left[A_0 R_0^{\frac{2n^2-3n+1}{n-2}} + \frac{I_3(n-1)\left(4n^2-10n+7\right)R_0^{3-2n}}{4f_0^2 n^3(n-2)\left(8n^2-19n+12\right)} \times \log\left(1 - \frac{2nr^{\frac{-8n^2+19n-12}{4n^2-10n+7}}}{(n-1)R_0}\right)\right] + \frac{I_3\left(4n^2-10n+7\right)R_0^{2-2n}}{2f_0^2 n^2(n-2)\left(8n^2-19n+12\right)} r^{\frac{-6n^2+16n-11}{4n^2-10n+7}}.$$

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Case I: $f(R,\phi,X) = f_0 R^n$

- We found also other solutions with $M(r) = r^q$ and $R(r) = R_0 r^{-q}$:

Analytical solution 3 ($n = \frac{1}{2}, q = \frac{2}{3}, R_0 = 1$)

$$A(r) = A_0\left(1 - \frac{2k}{r^{2/3}}\right), \quad B(r) = \frac{2}{21 r^{4/3}\left(1 - \frac{2k}{r^{2/3}}\right)}. \tag{20}$$

Analytical solution 4 ($q = \frac{2}{43}, R_0 = \frac{1006}{321}, n = 3$)

$$A(r) = \frac{A_0 R_0^{10}}{r^{20/43}}\left[1 + \frac{43 k R_0^{14}}{13(R_0 - 3)} r^{13/43}\right], B(r) = -\frac{r^{-84/43}}{86\left(R_0 - 3 + \frac{43}{13} k R_0^{14} r^{13/43}\right)}. \tag{21}$$

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Case I: $f(R,\phi,X) = f_0 R^2$ (specific subcase $n = 2$)

- For $M(r) = r^2$ and $f(R,\phi,X) = f_0 R^2$ (quadratic gravity):

Analytical solution 5

$$A = A_0\left(1+\frac{q}{r}\right)^{1-\frac{4I_3}{\alpha}}\left[R_0 r^2\left(1+\frac{q}{r}\right)^{\frac{2I_3}{\alpha}} - 4\right]^{-1}, \tag{22}$$

$$B = \frac{4}{\left(1+\frac{q}{r}\right)}\left[4 - R_0 r^2\left(1+\frac{q}{r}\right)^{\frac{2I_3}{\alpha}}\right]^{-1}. \tag{23}$$

- This solution is asymptotically flat if $0 < I_3/\alpha < 1/4$ and can describe a black hole since its horizons are at $r = -q$ and when $r^2 R_0(q/r+1)^{2I_3/\alpha} = 4$.

Case I: $f(R,\phi,X)=f_0R^2$ (specific subcase $n=2$)

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Case II: $f(R, \phi, X) = f_0 R + f_1 X^q - V(\phi)$

- For $M(r) = r^2$ we got:

Analytical solution 6 ($q = 1$ and $V = V_0 (\phi + V_1)^4$)

$$A = -\frac{1}{2 f_0 F_0 r} (I_1 r - I_2)^{I_{13}} \ , \ B = \frac{2 f_0 I_1 r}{F_0} (I_1 r - I_2)^{-2+I_{13}} \ , \tag{24}$$

$$\phi = \frac{1}{V_0 \sqrt{r}} \left[2 f_0 - F_0 (I_1 r - I_2)^{1-I_{13}} \right]^{\frac{1}{4}} - V_1 \ . \tag{25}$$

- If $I_{13} < 1$, the above metric is asymptotically flat and has an horizon at $r = I_2 / I_1$ with $I_{13} \neq 0$

Case II: $f(R,\phi,X) = f_0 R + f_1 X^q - V(\phi)$

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$$A = -\frac{1}{2 f_0 F_0 r}(I_1 r - I_2)^{I_{13}} \;, \; B = \frac{2 f_0 I_1 r}{F_0}(I_1 r - I_2)^{-2+I_{13}} \;, \tag{24}$$

$$\phi = \frac{1}{V_0 \sqrt{r}}\left[2 f_0 - F_0 (I_1 r - I_2)^{1-I_{13}}\right]^{\frac{1}{4}} - V_1 \,. \tag{25}$$

- If $I_{13} < 1$, the above metric is asymptotically flat and has an horizon at $r = I_2/I_1$ with $I_{13} \neq 0$

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Analytical solution 7 ($q = 2$ and $V = V_1 e^{-V_0\phi}$)

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$$B(r) = -4f_0^2 I_1 (I_1 r - I_2)^{-2\left(\frac{I_3}{I_1}+1\right)}\left(A_1 + \frac{A_0 r}{I_1}\right),,$$

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$$A(r) = A_0+\frac{(A_0I_2-4A_1I_3^2)}{2I_3r}-\frac{A_1I_2}{r^2}, \quad B(r)=4f_0^2\left(\frac{A_0r-2A_1I_3}{I_2+2I_3r}\right)$$

- This metric is asymptotically flat and has two horizons at the surfaces $r_1=2A_1I_3/A_0$ and $r_2=-I_2/(2I_3)$, i.e $A(r)=0$ at these surfaces.
- This metric behaves as Schwarzschild and additional contribution ($A_0=1, A_1=0, I_2=-4mI_3$ and $I_3=2f_0^2$ one gets Schwarzschild).

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Case III: $f(R,\phi,X) = f_0\phi^m R^n + f_1 X^q$

- For this case, taking $n = q = 1$:

Analytical solutions 8

$$A(r) = \frac{A_0}{C_1 I_1}(I_1 r - I_3)\,, \quad B(r) = \frac{A_0 I_1^2 C_1^{1-2m} r^{2m}}{4(I_3 - I_1 r)}\,, \quad \phi(r)^m = \frac{C_1}{r} \tag{26}$$

$$A(r) = \frac{A_0}{C_1} r\,, \quad B(r) = -\frac{A_0 I_1 C_1 r}{4\left(C_1 + \frac{\phi_0}{r}\right)^2}\,, \quad \phi(r)^m = \frac{C_1}{r} + \frac{\phi_0}{r^2} \tag{27}$$

- These are also new spherically symmetric solutions in this non-minimally couple theory between the scalar field and the Ricci scalar.
- They are non-asymptotically flat, and they have horizons at $r = I_3/I_1$ and $r = 0$, respectively.

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Case IV: $f(R, \phi, X) = U(\phi, X)\, R$.

Taking $U = f_0 X^q W(\phi)$ we found

Analytical solution 9 ($q = 1/4, M(r) = r^2$ and $R = \alpha/r^2$)

$$A = \frac{A_0}{K_1 r}, \quad B = \frac{1}{K_1 + \frac{K_2}{r}}, \tag{28}$$

Case IV: $f(R,\phi,X)=U(\phi,X)\,R$.

- Taking $U=f_0X^{1/2}\frac{dV(\phi)}{d\phi}$

Analytical solution 10 ($M(r)=r^2$ and $V=V_0r-V_1$)

$$A=\frac{\epsilon\left[2K_0-\frac{V_1I_1}{V_0}+(I_1+3V_0I_3)r+6V_0B_1I_1r^2\right]^2}{2I_1f_0^2(V_0r-V_1)^2}, \tag{29}$$

$$B=\frac{V_0\left[2K_0-\frac{V_1I_1}{V_0}+(I_1+3V_0I_3)r+6V_0B_1I_1r^2\right]}{4(V_0r-V_1)\left(B_1I_1+\frac{I_3}{2r}\right)}, \tag{30}$$

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$$r_\pm=\frac{1}{12V_0B_1I_1}\left[-(I_1+3V_0I_3)\pm\sqrt{(24B_1V_1+1)I_1^2+6V_0I_1(I_3-8B_1K_0)+9V_0^2I_3^2}\right], \tag{31}$$

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Conclusions

- New exact spherically symmetric solutions in extended $f(R, \phi, X)$ gravity were found.
- Some of these solutions can represent new black holes solutions in this extended theory of gravity
- The Noether symmetry in $f(R, \phi, X)$ gravity yields the solvable first integral of motion. With the conservation relation obtained from the Noether symmetry, the exact solutions for the field equations can be found.

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