Black holes solutions in metric-affine gravity with dynamical torsion and nonmetricity

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JCAP **01** (2022) no.01, 011; JCAP **04** (2022) no.04, 011; Jointly with Jorge Gigante Valcarcel.

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General Relativity - Assumptions

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- Strong gravity regime needs to be tested;
- A good way to understand GR is to modify it;

How to modify it?

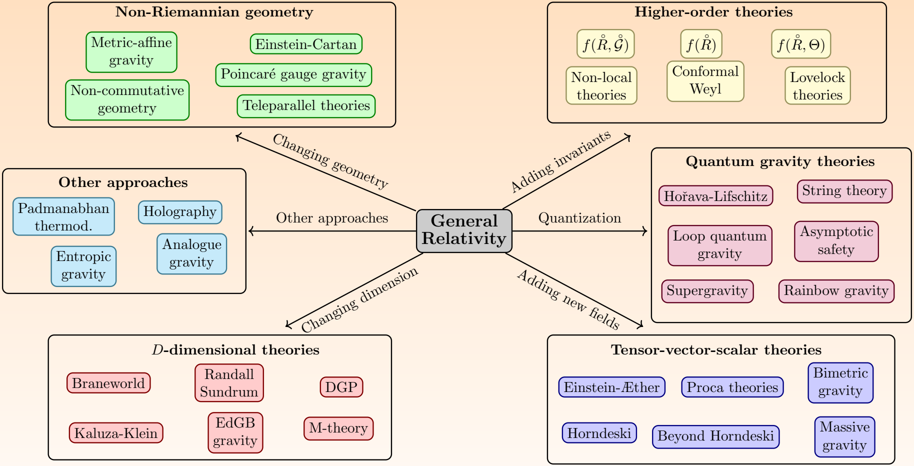

Figure: Classification of theories of gravity. (S. Bahamonde et.al., “Teleparallel Gravity: From Theory to Cosmology,” [arXiv:2106.13793 [gr-qc]].)

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Fundamental variables and characteristic tensors

- In the most general metric-affine setting, the fundamental variables are a **metric** $g_{\mu\nu}$ (10 comp.) as well as the coefficients $\tilde{\Gamma}^{\rho}{}_{\mu\nu}$ (64 comp.) of an **affine connection**.

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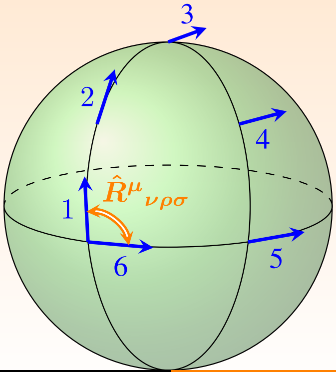
Curvature	$\tilde{R}^{\mu}{}_{\nu\rho\sigma} = \partial_{\rho}\tilde{\Gamma}^{\mu}{}_{\nu\sigma} - \partial_{\sigma}\tilde{\Gamma}^{\mu}{}_{\nu\rho} + \tilde{\Gamma}^{\mu}{}_{\tau\rho}\tilde{\Gamma}^{\tau}{}_{\nu\sigma} - \tilde{\Gamma}^{\mu}{}_{\tau\sigma}\tilde{\Gamma}^{\tau}{}_{\nu\rho}$
Torsion	$\tilde{T}^{\mu}{}_{\nu\rho} = \tilde{\Gamma}^{\mu}{}_{\rho\nu} - \tilde{\Gamma}^{\mu}{}_{\nu\rho}$
Nonmetricity	$\tilde{Q}_{\mu\nu\rho} = \tilde{\nabla}_{\mu}g_{\nu\rho} = \partial_{\mu}g_{\nu\rho} - \tilde{\Gamma}^{\sigma}{}_{\nu\mu}g_{\sigma\rho} - \tilde{\Gamma}^{\sigma}{}_{\rho\mu}g_{\nu\sigma}$

What does curvature geometrically represent?

Curvature tensor $\tilde{R}^{\alpha}{}_{\beta\mu\nu}$

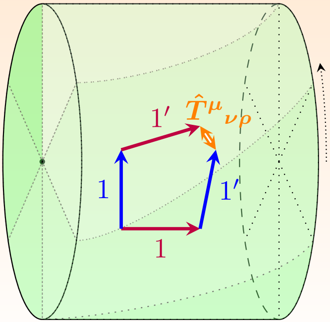
Rotation experienced by a vector when it is parallel transported along a closed curve



What does torsion geometrically represent?

Torsion tensor $\tilde{T}^{\alpha}{}_{\mu\nu}$

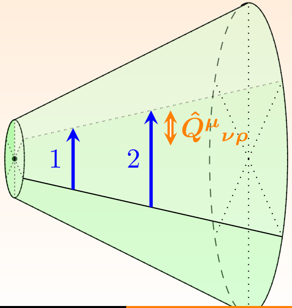
non-closure of the parallelogram formed when two infinitesimal vectors are parallel transported along each other.



What does non-metricity geometrically represent?

Non-metricity tensor $\tilde{Q}_{\alpha\mu\nu}$

measures how much the length and angle of vectors change as we parallel transport them, so in metric spaces the length of vectors is conserve



Some important special cases of metric-affine geometries

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- **General Teleparallel geometry** ($\tilde{R}_{\alpha\mu\nu\beta} = 0$): In the case of vanishing curvature, the connection is flat.

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- **Symmetric Teleparallel geometry** ($\tilde{R}_{\alpha\mu\nu\beta} = 0, \tilde{T}^{\alpha}_{\mu\nu} = 0$): Both torsion tensor and curvature are zero and the gravitational interactions are only mediated through non-metricity.

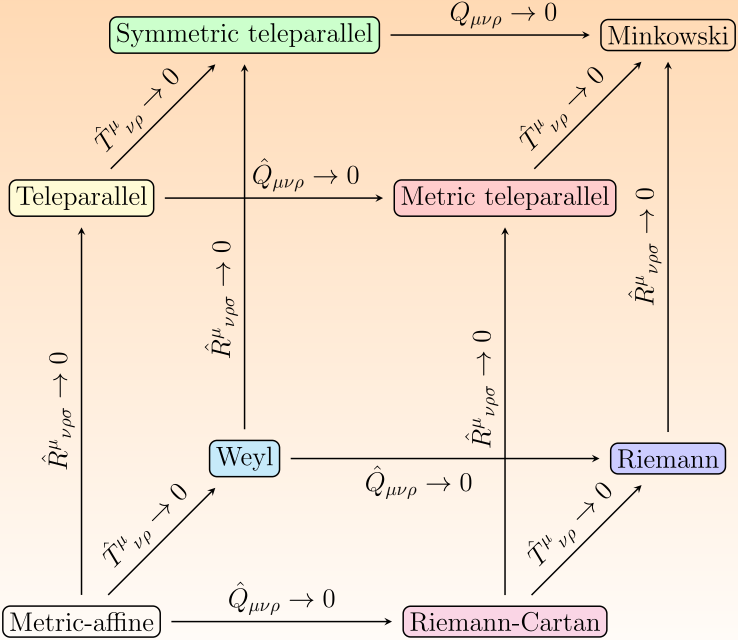
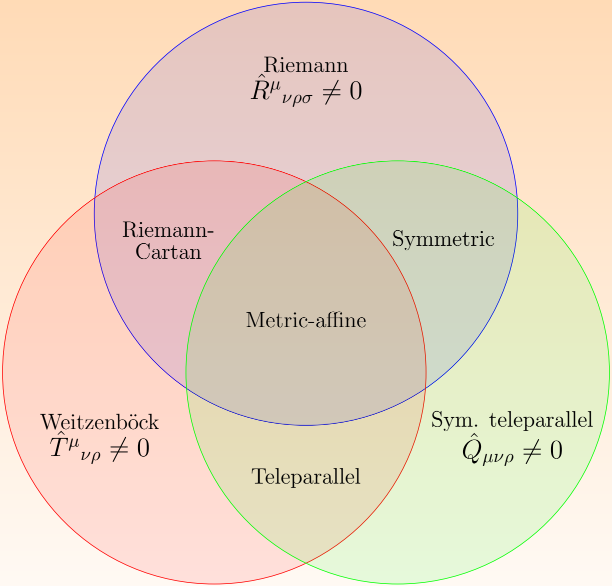


Figure: Classification of metric-affine geometries - Cube



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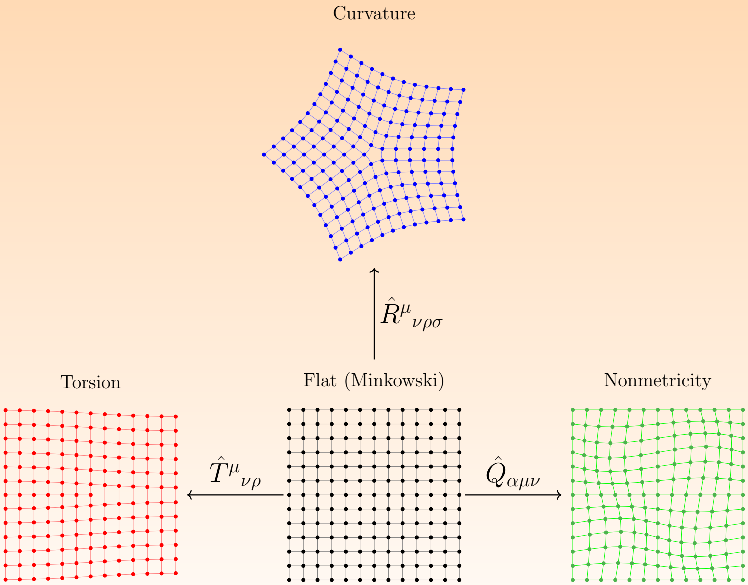

Figure: Crystalline structure and its analogy with curvature, torsion and non-metricity

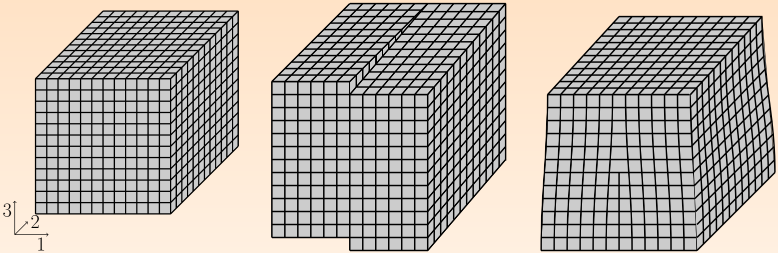

Figure: Crystal dislocations are shown against a regular crystal with no dislocation (left), and where *screw* (middle) and *edge* (right) dislocations are represented.

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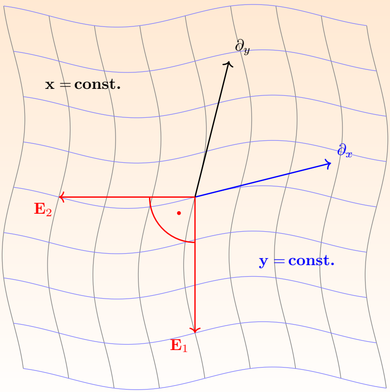
where $\eta_{ab} = \mathrm{diag}(1, -1, -1, -1)$ denotes the Minkowski metric.

Tetrads fields

The vectors $(\mathbf{E}_A)$ form an orthonormal basis of the tangent space, i.e.,

$$g(\mathbf{E}_a, \mathbf{E}_b) = g_{\mu\nu} {E_a}^\mu {E_b}^\nu = \eta_{ab} \,. \tag{2}$$



Spin connection and tetrads

- The frame coefficients $E_a{}^\mu$ are also required in order to calculate the coefficients $\tilde{\Gamma}^\mu{}_{\nu\rho}$ of the affine connection from the spin connection $\tilde{\omega}^a{}_{b\mu}$ via

$$\tilde{\Gamma}^\rho{}_{\mu\nu} = E_a{}^\rho \left(\partial_\nu e^a{}_\mu + \tilde{\omega}^a{}_{b\nu} e^b{}_\mu\right) \,, \tag{3}$$

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- This is the unique affine connection satisfying the so-called "tetrad postulate"

$$\partial_\mu e^a{}_\nu + \tilde{\omega}^a{}_{b\mu} e^b{}_\nu - \tilde{\Gamma}^\rho{}_{\nu\mu} e^a{}_\rho = 0 . \quad (4)$$

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- Then, we can define the curvature, torsion and nonmetricity as:

$$\tilde{R}^a{}_{b\mu\nu} := \partial_\mu \tilde{\omega}^a{}_{b\nu} - \partial_\nu \tilde{\omega}^a{}_{b\mu} + \tilde{\omega}^a{}_{c\mu}\tilde{\omega}^c{}_{b\nu} - \tilde{\omega}^a{}_{c\nu}\tilde{\omega}^c{}_{b\mu}\,, \quad (5)$$

$$\tilde{T}^a{}_{\mu\nu} := \partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu + \tilde{\omega}^a{}_{b\mu} e^b{}_\nu - \tilde{\omega}^a{}_{b\nu} e^b{}_\mu\,, \quad (6)$$

$$\tilde{Q}_{\mu ab} := -\eta_{ac}\tilde{\omega}^c{}_{b\mu} - \eta_{cb}\tilde{\omega}^c{}_{a\mu}\,. \quad (7)$$

Outline

Trinity of gravity

As mentioned before, we can split the connection as

$$\tilde{\Gamma}^{\rho}{}_{\mu\nu} := \Gamma^{\rho}{}_{\mu\nu} + \tilde{K}^{\rho}{}_{\mu\nu} + \tilde{L}^{\rho}{}_{\mu\nu} := \Gamma^{\rho}{}_{\mu\nu} + \tilde{D}^{\rho}{}_{\mu\nu}\,, \qquad (8)$$

where

$$\Gamma^{\mu}{}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} \left(\partial_\nu g_{\sigma\rho} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho} \right) , \quad \text{Levi Civita connection}$$

$$\tilde{K}^{\mu}{}_{\nu\rho} = \frac{1}{2} \left(\tilde{T}_{\nu}{}^{\mu}{}_{\rho} + \tilde{T}_{\rho}{}^{\mu}{}_{\nu} - \tilde{T}^{\mu}{}_{\nu\rho} \right) , \quad \text{Contortion tensor}$$

$$\tilde{L}^{\mu}{}_{\nu\rho} = \frac{1}{2} \left(\tilde{Q}^{\mu}{}_{\nu\rho} - \tilde{Q}_{\nu}{}^{\mu}{}_{\rho} - \tilde{Q}_{\rho}{}^{\mu}{}_{\nu} \right) , \quad \text{Disformation tensor}\,.$$

Trinity of gravity - curvature tensor

- The curvature becomes

$$\tilde{R}^{\mu}{}_{\nu\rho\sigma} = R^{\mu}{}_{\nu\rho\sigma} + \mathring{\nabla}_{\rho}\tilde{D}^{\mu}{}_{\nu\sigma} - \mathring{\nabla}_{\sigma}\tilde{D}^{\mu}{}_{\nu\rho} + \tilde{D}^{\mu}{}_{\tau\rho}\tilde{D}^{\tau}{}_{\nu\sigma} - \tilde{D}^{\mu}{}_{\tau\sigma}\tilde{D}^{\tau}{}_{\nu\rho}\,.$$

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Ricci scalar decomposition

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with

$$T := T^{\rho\lambda\kappa}T_{\rho\lambda\kappa} + 2T^{\rho\lambda\kappa}T_{\kappa\rho\lambda} - 4T_{\rho}{}^{\kappa}{}_{\kappa}T^{\rho\lambda}{}_{\lambda}\,, \quad \text{Torsion scalar}\,,$$

$$Q := -\frac{1}{4}Q_{\alpha\beta\gamma}Q^{\alpha\beta\gamma} + \frac{1}{2}Q_{\alpha\beta\gamma}Q^{\beta\alpha\gamma} + \frac{1}{4}Q_{\alpha}Q^{\alpha} - \frac{1}{2}Q_{\alpha}\bar{Q}^{\alpha}\,, \ \text{Nonmetricity scalar}$$

$$C := 2(Q_{\kappa\rho\lambda}T^{\lambda\kappa\rho} + Q_{\rho}{}^{\sigma}{}_{\sigma}T^{\rho\kappa}{}_{\kappa} - Q^{\sigma}{}_{\sigma\rho}T^{\rho\kappa}{}_{\kappa})\,.$$

Trinity of gravity - General Relativity

- GR assumes zero torsion and nonmetricity so that

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where $\kappa^2 = 8\pi G$ and $L_{\rm m}$ is any matter Lagrangian.

- The Einstein's field equations are obtained by taking variations w/r to the metric: $\boxed{R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} = \kappa^2 T_{\mu\nu}}$.

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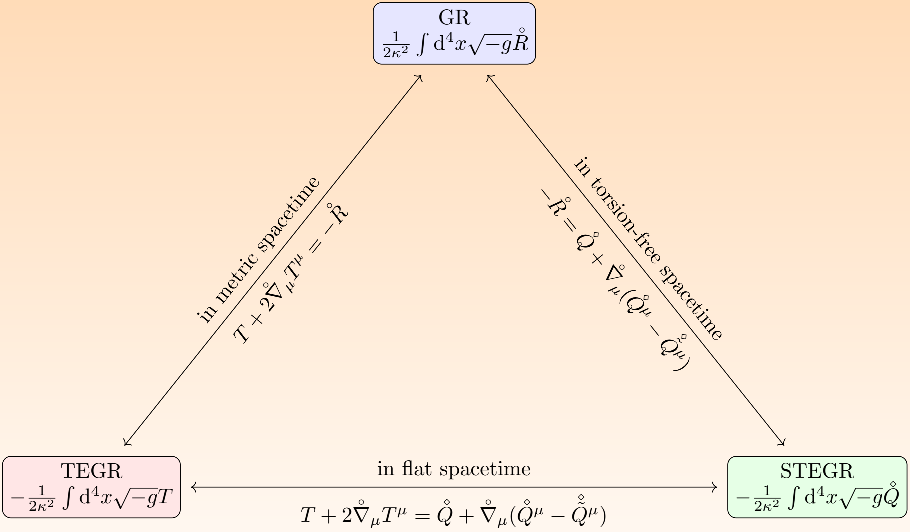

Figure: Geometrical trinity of gravity (S. Bahamonde et.al., "Teleparallel Gravity: From Theory to Cosmology," [arXiv:2106.13793 [gr-qc]].)

Outline

Gauge formalism of Poincaré gauge gravity

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- Not only an energy-momentum tensor of matter arises, but also a nontrivial spin density tensor which operates as source of torsion$\Longrightarrow$ an extended correspondence between the geometry of the space-time and the properties of matter.

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- Affine group $A(4,\mathcal{R}) = R^4 \otimes GL(4,R)$ is the semiproduct of the translation group R^4 and the general linear group $GL(4,R)$. gauge connection with an independent local metric structure[1]:

$$A_\mu = e^a{}_\mu P_a + \omega^a{}_{b\mu} L_a{}^b\,, \tag{9}$$

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- Generators of the group $A(4,\mathcal{R})$:

$$[P_a, P_b] = 0 \,, \tag{11}$$

$$\left[L_a{}^b, P_c\right] = i\,\delta^b{}_c\,P_a \,, \tag{12}$$

$$\left[L_a{}^b, L_c{}^d\right] = i\left(\delta^b{}_c\,L_a{}^d - \delta_a{}^d\,L_c{}^b\right) . \tag{13}$$

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- it is possible to obtain the following gauge curvatures from the anholonomic metric, coframe and connection:

$$G_{ab\mu} = \partial_\mu g_{ab} - g_{ac}\,\omega^c{}_{b\mu} - g_{bc}\,\omega^c{}_{a\mu}\,, \tag{14}$$

$$F^a{}_{\mu\nu} = \partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu + \omega^a{}_{b\mu}\,e^b{}_\nu - \omega^a{}_{b\nu}\,e^b{}_\mu\,, \tag{15}$$

$$F^a{}_{b\mu\nu} = \partial_\mu \omega^a{}_{b\nu} - \partial_\nu \omega^a{}_{b\mu} + \omega^a{}_{c\mu}\,\omega^c{}_{b\,\nu} - \omega^a{}_{c\nu}\,\omega^c{}_{b\mu}\,. \tag{16}$$

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- Correspondence with the curvature, torsion and nonmetricity tensors:

$$G_{ab\mu} = g_{ac}g_{bd}e^{c\lambda}e^{d\rho}Q_{\mu\lambda\rho}, \quad (17)$$

$$F^a{}_{\mu\nu} = e^a{}_\lambda T^\lambda{}_{\nu\mu}\,, \quad (18)$$

$$F^a{}_{b\mu\nu} = g_{bc}\,e^a{}_\lambda e^{c\rho}\tilde{R}^\lambda{}_{\rho\mu\nu}\,. \quad (19)$$

Outline

Dynamics of metric-affine geometry

- Gravitational action with dynamical torsion and nonmetricity:

$$S = \int d^4x\sqrt{-g}\left[\mathcal{L}_m - \frac{1}{16\pi}\mathcal{L}_g(\tilde{\mathcal{R}}, \mathcal{T}, \mathcal{Q})\right]. \tag{20}$$

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Here $\theta_a{}^\nu$ is the energy-momentum tensor (canonical) and $\Delta_a{}^{b\nu}$ is the hypermomentum density tensor.

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- $GL(4,R)$ group allows the definition of a large number of scalar invariants depending on the aforementioned tensors.

Dynamics of metric-affine geometry

- General quadratic gravitational action with dynamical torsion and nonmetricity:

$$S=\int d^4x\sqrt{-g}\Big\{\mathcal{L}_m+\frac{1}{16\pi}\Big[-\tilde{R}+a_1\tilde{R}^2+a_2\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\lambda\rho\mu\nu}+a_3\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\rho\lambda\mu\nu}$$
$$+a_4\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\mu\nu\lambda\rho}+a_5\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\lambda\mu\rho\nu}+a_6\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\mu\lambda\rho\nu}+a_7\tilde{R}_{\rho\lambda\mu\nu}\tilde{R}^{\mu\lambda\rho\nu}$$
$$+a_8\tilde{R}_{\mu\nu}\tilde{R}^{\mu\nu}+a_9\tilde{R}_{\mu\nu}\tilde{R}^{\nu\mu}+a_{10}\hat{R}_{\mu\nu}\hat{R}^{\mu\nu}+a_{11}\hat{R}_{\mu\nu}\hat{R}^{\nu\mu}+a_{12}\tilde{R}_{\mu\nu}\hat{R}^{\mu\nu}$$
$$+a_{13}\tilde{R}_{\mu\nu}\hat{R}^{\nu\mu}+a_{14}\tilde{R}^{\lambda}{}_{\lambda\mu\nu}\tilde{R}^{\rho}{}_{\rho}{}^{\mu\nu}+a_{15}\tilde{R}^{\lambda}{}_{\lambda\mu\nu}\tilde{R}^{\mu\nu}+a_{16}\tilde{R}^{\lambda}{}_{\lambda\mu\nu}\hat{R}^{\mu\nu}$$
$$+b_1T_{\lambda\mu\nu}T^{\lambda\mu\nu}+b_2T_{\lambda\mu\nu}T^{\mu\lambda\nu}+b_3T^{\lambda}{}_{\lambda\nu}T^{\mu}{}_{\mu}{}^{\nu}+c_1T_{\lambda\mu\nu}Q^{\mu\lambda\nu}$$
$$+c_2T^{\lambda}{}_{\lambda\nu}Q^{\nu\mu}{}_{\mu}+c_3T^{\lambda}{}_{\lambda\nu}Q^{\mu\nu}{}_{\mu}+d_1Q_{\lambda\mu\nu}Q^{\lambda\mu\nu}+d_2Q_{\lambda\mu\nu}Q^{\mu\lambda\nu}$$
$$+d_3Q^{\lambda}{}_{\lambda\nu}Q^{\mu}{}_{\mu}{}^{\nu}+d_4Q_{\nu}{}^{\lambda}{}_{\lambda}Q^{\nu\mu}{}_{\mu}+d_5Q^{\lambda}{}_{\lambda\nu}Q^{\nu\mu}{}_{\mu}\Big]\Big\}\,. \tag{23}$$

MAG models with dynamical torsion and nonmetricity

- In order to have a theory such that when $T = Q = 0$ one recovers GR, one can relate the constants.

[2] S. Bahamonde and J. G. Valcarcel, JCAP **09**, 057 (2020).

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- Quadratic gravitational action with dynamical torsion and nonmetricity in Weyl-Cartan geometry ($Q_{\lambda\mu\nu} = g_{\mu\nu} W_\lambda$)

$$S = \int d^4x\sqrt{-g}\Big\{\mathcal{L}_{\rm m} + \frac{1}{64\pi}\Big[-4R - 6d_1\tilde{R}_{\lambda[\rho\mu\nu]}\tilde{R}^{\lambda[\rho\mu\nu]}$$

$$- 9d_1\tilde{R}_{\lambda[\rho\mu\nu]}\tilde{R}^{\mu[\lambda\nu\rho]} + 8\,d_1\tilde{R}_{[\mu\nu]}\tilde{R}^{[\mu\nu]} + \frac{1}{8}\left(32e_1 + 8e_2 + 17d_1\right)\tilde{R}^{\lambda}{}_{\lambda\mu\nu}\tilde{R}^{\rho}{}_{\rho}{}^{\mu\nu}$$

$$- 7d_1\tilde{R}_{[\mu\nu]}\tilde{R}^{\lambda}{}_{\lambda}{}^{\mu\nu} + 3\left(1 - 2a_2\right)T_{[\lambda\mu\nu]}T^{[\lambda\mu\nu]}\Big]\Big\}\,.$$

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- Absence of a general Birkhoff's theorem in MAG: new spherically and axially symmetric vacuum solutions with independent dynamical torsion and nonmetricity fields[2,3]

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Outline

Spherical symmetry

- Metric, torsion and nonmetricity in spherically symmetric space-times ($\#2 + \#8 + \#2 = \#12$):

$$\mathcal{L}_\xi g_{\mu\nu} = \mathcal{L}_\xi T^{\lambda}{}_{\mu\nu} = \mathcal{L}_\xi W_\mu = 0 \implies \mathcal{L}_\xi \tilde{R}_{\lambda\rho\mu\nu} = 0$$

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- By solving these equations we find that torsion and nonmetricity behave as

$$T^t{}_{tr} = a(r)\,, \quad T^r{}_{tr} = b(r)\,, \quad T^{\theta_k}{}_{t\theta_k} = f(r)\,, \quad T^{\theta_k}{}_{r\theta_k} = g(r)$$
$$T^{\theta_k}{}_{t\theta_l} = e^{a\theta_k}\, e^b{}_{\theta_l}\, \epsilon_{ab}\, d(r)\,, \quad T^{\theta_k}{}_{r\theta_l} = e^{a\theta_k}\, e^b{}_{\theta_l}\, \epsilon_{ab}\, h(r)\,,$$
$$T^t{}_{\theta_k\theta_l} = \epsilon_{kl}\, k(r)\, \sin\theta_1\,, \quad T^r{}_{\theta_k\theta_l} = \epsilon_{kl}\, l(r)\, \sin\theta_1\,,$$
$$W_\lambda = (w_1(r), w_2(r), 0, 0)\,,$$

whereas the metric is in the standard spherically symmetric form:

$$ds^2 = \Psi_1(r)\, dt^2 - \frac{dr^2}{\Psi_2(r)} - r^2\left(d\theta_1^2 + \sin^2\theta_1 d\theta_2^2\right)\,.$$

Here, ϵ_{kl} is the Levi-Civita symbol in two dimensions.

Spherical symmetry - Solving the field equations

The field eqs are very involved. To solve them we use the following strategy:

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   Regularity restricts the initial arbitrariness of the torsion components and the Weyl vector by imposing the relations

$$b(r) = a(r)\sqrt{\Psi_1(r)\Psi_2(r)}\,, \qquad f(r) = -g(r)\sqrt{\Psi_1(r)\Psi_2(r)}\,,$$
$$d(r) = -h(r)\sqrt{\Psi_1(r)\Psi_2(r)}\,, \quad l(r) = k(r)\sqrt{\Psi_1(r)\Psi_2(r)}\,,$$
$$w_1(r) = -w_2(r)\sqrt{\Psi_1(r)\Psi_2(r)}\,.$$

Spherical symmetry - Solving the field equations

2. **Solve the weak field limit:** The weak field limit of the field equations become

$$\nabla_\rho \nabla_\lambda T^{\lambda\rho}{}_{\mu} + \nabla_\rho \nabla^\rho T^{\lambda}{}_{\mu\lambda} - \nabla_\rho \nabla_\mu T^{\lambda\rho}{}_{\lambda} = 0\,,$$
$$\nabla_\mu \tilde{R}^{\lambda}{}_{\lambda}{}^{\mu\nu} = 0\,.$$

Spherical symmetry - Solving the field equations

2. **Solve the weak field limit:** The weak field limit of the field equations become

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These equations can be solved, yielding

$$w_1(r) = -\,\kappa_d \int \sqrt{\frac{\Psi_1(r)}{\Psi_2(r)}}\,\frac{dr}{r^2}\,,$$

$$b(r) = r f'(r) + f(r) + \frac{\kappa_d}{2r}\sqrt{\frac{\Psi_1(r)}{\Psi_2(r)}}\,,$$

where κ_d is an integration constant which represents the dilaton charge.

Spherical symmetry - Solving the field equations

1. The final solution for the metric behaves as Reissner-Nordström

$$g_{tt} = -1/g_{rr} \equiv \Psi(r) = 1 - \frac{2m}{r} + \frac{d_1\kappa_{\mathrm{s}}^2 - 4e_1\kappa_{\mathrm{d,e}}^2}{r^2}\,. \qquad (24)$$

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$$W_\mu = \frac{\kappa_{\rm d,e}}{r}\left(1, -1/\Psi(r), 0, 0\right)\,. \quad (25)$$

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3. Torsion sector:

$$\bar{\mathcal{S}}^a = -\frac{\kappa_{\rm s}}{r}(1,1,0,0)\,, \tag{26}$$

$$\bar{\mathcal{T}}_2^{abc} = \frac{\kappa_{\rm s}}{3r} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} . \tag{27}$$

Dilation and spin charges

What do κ_s (dilation charge) and $\kappa_{d,e}$ (spin charge) physically represent?

Point 1 - Hypermomentum density

In the geometric scheme of MAG, not only an energy-momentum tensor of matter arises as source of curvature, but also a hypermomentum density tensor which operates as source of torsion and nonmetricity.

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Point 2 - Dilation and spin charges

In Weyl-Cartan geometry, hypermomentum density tensor splits into spin and dilation currents, which carry their own charges and provide a RN solution.

Dilation and spin charges

When these charges might be important?

Significant effects are contemplated only around **extreme gravitational systems**, such as **neutron stars** with intense magnetic fields and sufficiently oriented elementary spins or **black holes** endowed with spin and dilation charges.

Outline

Nature of the black hole solution

- The nature of the horizons depends on the difference $d_1\kappa_s^2 - 4e_1\kappa_d^2$. Thus, a positive difference of this quantity would present two horizons determined from the roots

$$r_{\pm} = M \pm \Delta_1\,, \quad \Delta_1^2 = M^2 - \left(d_1\kappa_s^2 - 4e_1\kappa_d^2\right)\,,$$

with $0 < \left(d_1\kappa_s^2 - 4e_1\kappa_d^2\right) < M^2$.

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- The different signs of the kinetic terms related to the dynamical part of torsion and to the Weyl vector allows the case $d_1\kappa_s^2 - 4e_1\kappa_d^2 < 0$.
- The balance between κ_d and κ_s **is not restricted** to any special constraint and therefore any of these situations may occur in the presence of torsion and nonmetricity.

Particle motion in MAG

- The equations of motion of test bodies with microstructure coupled to the torsion and nonmetricity tensors become[4]

$$\dot{p}^{\mu} + \Gamma^{\mu}{}_{\lambda\rho}\, p^{\lambda} u^{\rho} + N_{[\lambda\rho]}{}^{\mu} p^{\rho} u^{\lambda} + \tilde{R}_{\lambda\rho\sigma}{}^{\mu} \triangle^{\rho\lambda} u^{\sigma} = 0\,.$$

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- This eq. reduces to the standard geodesic one ($\dot{p}^{\mu} + \Gamma^{\mu}{}_{\lambda\rho}\, p^{\lambda} u^{\rho} = 0$) when the hypermomentum of the test body vanishes and also when the particle are bosons.
- Using the standard geodesic approach, we find

$$\frac{1}{2}\dot{r}^2 + V(r) = 0\,, \quad V(r) = -\frac{1}{2}c^2 E^2 + \frac{1}{2}\Psi(r)\left(\frac{J^2}{r^2} + \sigma c^2\right)\,,$$

where E an J are the conserved charges and $\sigma = 0(\sigma = 1)$ represents massless(massive) particles.

[4] D. Puetzfeld and Y. N. Obukhov, Phys. Rev. D **76** (2007), 084025.

Observational constrains

- Let us now consider the case where the effect of torsion dominates over the contribution of nonmetricity.

[5] S. Bahamonde and J. Gigante Valcarcel, Eur. Phys. J. C **81** (2021) no.6, 495.

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- Let us now consider the case where the effect of torsion dominates over the contribution of nonmetricity.
- Indeed, due to the presence of a magnetic field in white dwarfs, it is expected that Sirius B can have sufficiently oriented elementary spins in comparison with an effective dilation charge, therefore, $\kappa_{s,\mathrm{SiriusB}} \gg \kappa_{d,\mathrm{SiriusB}}$.

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- **Perihelion shift+ Gravitational redshift:** Assuming the same approximation in Sgr A* and considering the universality of the coupling constant d_1, we find[5]

$$1.396 \cdot 10^{10} \leq \frac{\kappa_{s,\mathrm{SgrA}*}}{\kappa_{s,\mathrm{SiriusB}}} \leq 1.688 \cdot 10^{10}\,.$$

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- To the best of our knowledge, this bound provides the first observational comparison between the spin charges of a supermassive black hole and a degenerate star.

[5] S. Bahamonde and J. Gigante Valcarcel, Eur. Phys. J. C **81** (2021) no.6, 495.

Outline

Extension to axisymmetric space-times

- Metric, torsion and nonmetricity tensors in symmetric space-times:

$$\mathcal{L}_\xi g_{\mu\nu} = \mathcal{L}_\xi T^\lambda{}_{\mu\nu} = \mathcal{L}_\xi Q^\lambda{}_{\mu\nu} = 0 \implies \mathcal{L}_\xi \tilde{R}^\lambda{}_{\rho\mu\nu} = 0\,. \quad (28)$$

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- Stationary and axisymmetric space-times:

$$\#10 \to \#4 \begin{cases} ds^2 = \Psi_1(r,\vartheta)\,dt^2 - \frac{dr^2}{\Psi_2(r,\vartheta)} \\ -r^2\Psi_3(r,\vartheta)\left[d\vartheta^2 + \sin^2\vartheta(d\varphi - \Psi_4(r,\vartheta)dt)^2\right] \end{cases} ;$$

$$\#24 \left\{ T^\lambda{}_{\mu\nu} = T^\lambda{}_{\mu\nu}(r,\vartheta) \right. \quad (29)$$

$$\#4 \left\{ W_\mu = (W_t(r,\vartheta), W_r(r,\vartheta), W_\vartheta(r,\vartheta), W_\varphi(r,\vartheta))\,. \right. \quad (30)$$

Axisymmetric space-times - Kerr-Newmann de-Sitter

- Rotating Kerr-Newman metric structure[6]:

$$
\begin{aligned}
ds^2 = {} & \Psi(r,\vartheta)\, dt^2 - \frac{r^2 + a^2\cos^2\vartheta}{(r^2 + a^2\cos^2\vartheta)\,\Psi(r,\vartheta) + a^2\sin^2\vartheta}\, dr^2 \\
& - \left(r^2 + a^2\cos^2\vartheta\right) d\vartheta^2 + 2a\,(1 - \Psi(r,\vartheta))\sin^2\vartheta\, dt d\varphi \\
& - \sin^2\vartheta\left[r^2 + a^2 + a^2\,(1 - \Psi(r,\vartheta))\sin^2\vartheta\right] d\varphi^2\,,
\end{aligned} \tag{31}
$$

$$
\Psi(r,\vartheta) = 1 - \frac{\left[2mr + 4e_1(\kappa^2_{\rm d,e} + \kappa^2_{\rm d,m}) - d_1\kappa^2_{\rm s}\right]}{r^2 + a^2\cos^2\vartheta}\,. \tag{32}
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[6] S. Bahamonde and J. G. Valcarcel, JCAP **01** (2022) no.01, 011.

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- Field strength tensors:

$$\begin{aligned} \bar{R}_{[\mu\nu]} &= \frac{1}{12}\varepsilon^{\lambda}{}_{\sigma\mu\nu}\nabla_\lambda \bar{S}^\sigma + \frac{1}{2}\nabla_\lambda \bar{t}^{\lambda}{}_{\mu\nu}\,; \quad \tilde{R}^{\lambda}{}_{\lambda\mu\nu} = 4\nabla_{[\nu}W_{\mu]}\,; \\ \bar{R}^{\lambda}{}_{[\mu\nu\rho]} &= \frac{1}{6}\varepsilon^{\lambda}{}_{\sigma[\rho\nu}\nabla_{\mu]}\bar{S}^\sigma + \nabla_{[\mu}\bar{t}^{\lambda}{}_{\rho\nu]} + \frac{1}{4}\varepsilon^{\lambda}{}_{\omega\sigma[\rho}\mathring{t}^{\sigma}_{1}{}_{\mu\nu]}\bar{S}^\omega \\ &\quad - \frac{1}{18}\varepsilon_{\sigma\mu\nu\rho}\mathring{T}^{\lambda}_{1}\bar{S}^\sigma\,. \end{aligned} \tag{33}$$

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- Nonmetricity sector:(no approx.)

$$w_1(r,\vartheta) = \frac{\kappa_{\rm d,e} r - a\,\kappa_{\rm d,m}\cos\vartheta}{r^2 + a^2\cos^2\vartheta}\,, \quad w_3(r,\vartheta) = 0\,,$$
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$$\bar{\mathcal{T}}_2^{abc} = \frac{\kappa_{\mathrm{s}}}{3r}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 \end{pmatrix} + \mathcal{O}(a\kappa_{\mathrm{s}})\,. \quad (36)$$

Gravitational spin-orbit interaction

- We found a solution in the decoupling limit $a\kappa_s \ll 1$, which ensures that the Maxwell equation and closure conditions are fulfilled by the field strength tensors of torsion

$$\nabla_\lambda \tilde{R}^\lambda{}_{[\rho\mu\nu]} \;=\; \nabla_\mu \tilde{R}^{[\mu\nu]} = 0\,, \quad \nabla_{[\sigma}\tilde{R}_{\lambda[\rho\mu\nu]]} = \nabla_{[\lambda}\tilde{R}_{[\mu\nu]]} = 0\,.$$

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The dynamics of torsion and nonmetricity alters the geometry of the space-time $\Longrightarrow$

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- **Gravitational spin-orbit interaction:**

$$\mathcal{H}_{\rm LS} = \frac{1}{m_{\rm e}^2 r}\frac{\partial V}{\partial r}\mathbf{L}\cdot\mathbf{S} \approx \frac{d_1}{2r}\frac{\partial g_{tt}}{\partial r}a\kappa_{\rm s}\cos\vartheta \qquad (37)$$

Extension to axisymmetric space-times - Plebanski-Damianski

- It is well known that the most general axisymmetric system in vacuum that can describe a BH type D in **GR** contains[7]:

Mass	M
Angular momentum	a
Taub-NUT charge	l
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[7] J. F. Plebanski and M. Demianski, Annals Phys. **98** (1976), 98-127

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- The solution in GR is called Plebanski-Damianski solution.

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- The Plebanski-Damianski metric was recently presented in an improved form with $\Lambda = 0$ in by Podolský and Vrátný (Phys. Rev. D **104** (2021), 084078), and it can be written as

$$ds^2 = \Omega^{-2}(r,\vartheta)\Big\{\Phi_1(r,\vartheta)\left[dt - \left(a\sin^2\vartheta + 2l(\chi - \cos\vartheta)\right)d\varphi\right]^2 - \frac{dr^2}{\Phi_1(r,\vartheta)} - \frac{d\vartheta^2}{\Phi_2(r,\vartheta)} - \Phi_2(r,\vartheta)\sin^2\vartheta\left[a\,dt - \left(r^2 + a^2 + l^2 + 2\chi a l\right)d\varphi\right]^2\Big\}.$$

where Φ_i, Ω are cumbersome functions depending on these parameters.

[8] S. Bahamonde, J. G. Valcarcel and L. Järv, JCAP **04** (2022) no.04, 011.

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  where Φ_i, Ω are cumbersome functions depending on these parameters.
- We just found this new form with the cosmological constant[8] with $\Phi_1(r,\vartheta) = \frac{Q(r)}{\rho^2(r,\vartheta)}$, $\quad \Phi_2(r,\vartheta) = \frac{P(\vartheta)}{\rho^2(r,\vartheta)}$, and $\rho^2(r,\vartheta) = r^2 + (a\cos\vartheta + l)^2$. Here, $Q(r), \Omega(\vartheta)$ include the PD quantities.

[8] S. Bahamonde, J. G. Valcarcel and L. Järv, JCAP **04** (2022) no.04, 011.

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- We found a solution to **OUR THEORY** in the decoupling limit $|x_i \kappa_s| \ll 1$ with $x = (a, l, \alpha)$ with additional torsion and nonmetricity terms

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$$w_3(r,\vartheta) = -\kappa_{\rm d,m}\sqrt{K(\vartheta) - \left(\frac{\cot\vartheta - \gamma\csc\vartheta}{P(\vartheta)}\right)^2}\,,$$

$$w_4(r,\vartheta) = \kappa_{\rm d,m}\left[\frac{\left(r^2 + a^2 - l^2\right)\cos\vartheta + al\sin^2\vartheta + 2\chi l\,(a\cos\vartheta + l)}{r^2 + (a\cos\vartheta + l)^2} - \gamma\right] - \frac{\kappa_{\rm d,e} r\left[a\sin^2\vartheta + 2l\,(\chi - \cos\vartheta)\right]}{r^2 + (a\cos\vartheta + l)^2}\,,$$

$$\bar{T}^{\vartheta}{}_{\varphi t} = -\bar{T}^{\varphi}{}_{\vartheta t}\sin^2\vartheta = -\bar{T}^{\vartheta}{}_{\varphi r}\frac{Q(r)}{\rho^2(r,\vartheta)} = \bar{T}^{\varphi}{}_{\vartheta r}\frac{Q(r)}{\rho^2(r,\vartheta)}\sin^2\vartheta = \frac{\kappa_s\sin\vartheta}{r} + \mathcal{O}(x_i\kappa_s)\,.$$

- Similarly as electromagnetism, the torsion behaves as a Coulomb-like quantity depending on a spin charge κ_s and the non-metricity on the dilation charge κ_d.

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- Future: search of a gravitational spin-orbit interaction in MAG beyond the Kerr-Newman space-time (MAG is the main candidate to describe a spin-orbit interaction beyond GR).