Black holes solutions in metric-affine gravity with dynamical torsion and nonmetricity

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Outline

- Introduction to Metric-affine gravity
 - Why modified gravity?
 - Basic geometrical quantities
 - Tetrads and spin connection
- Trinity of gravity
 - Trinity of gravity: GR, TEGR and STEGR.
- 3 Metric-Affine gravity
 - Gauge formalism
 - Oynamics
- MAG models with dynamical torsion and nonmetricity
 - Spherical symmetry
 - Observational constraints
 - Axial symmetry

Introduction to Metric-affine gravity

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Why modified gravity? Basic geometrical quantities Tetrads and spin connection

General Relativity - Assumptions

General Relativity is based upon different assumptions that can be understood as the fulfilling of the Lovelock's theorem. Some assumptions are:

Equivalence principle

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Why modified gravity?

• GR is not compatible with quantum field theory;

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- Strong gravity regime needs to be tested;
- A good way to understand GR is to modify it;

How to modify it?



Figure: Classification of theories of gravity. (S. Bahamonde et.al., "Teleparallel Gravity: From Theory to Cosmology," [arXiv:2106.13793 [gr-qc]].)

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Why modified gravity? Basic geometrical quantities Tetrads and spin connection

Fundamental variables and characteristic tensors

In the most general metric-affine setting, the fundamental variables are a metric g_{μν} (10 comp.) as well as the coefficients Γ^ρ_{μν} (64 comp.) of an affine connection.

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What does curvature geometrically represent?

Curvature tensor $\tilde{R}^{\alpha}{}_{\beta\mu\nu}$

Rotation experienced by a vector when it is parallel transported along a closed curve



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What does torsion geometrically represent?

Torsion tensor $\tilde{T}^{\alpha}{}_{\mu\nu}$

non-closure of the parallelogram formed when two infinitesimal vectors are parallel transported along each other.



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MAG models with dynamical torsion and nonmetricity

What does non-metricity geometrically represent?

Non-metricity tensor $\tilde{Q}_{\alpha\mu\nu}$

measures how much the length and angle of vectors change as we parallel transport them, so in metric spaces the length of vectors is conserve



Why modified gravity? Basic geometrical quantities Tetrads and spin connection

Some important special cases of metric-affine geometries

• **Riemann-Cartan geometry** $(\tilde{Q}_{\alpha\mu\nu} = 0)$: If non-metricity vanishes, the metric satisfies the metric-compatibility condition $\tilde{\nabla}_{\mu}g_{\alpha\beta} = 0$. Poincaré grvity assumes this geometry.

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- General Teleparallel geometry ($\tilde{R}_{\alpha\mu\nu\beta} = 0$): In the case of vanishing curvature, the connection is flat.

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- Symmetric Teleparallel geometry ($\tilde{R}_{\alpha\mu\nu\beta} = 0, \tilde{T}^{\alpha}_{\mu\nu} = 0$): Both torsion tensor and curvature are zero and the gravitational interactions are only mediated through non-metricity.



Figure: Classification of metric-affine geometries - Cube



Figure: Classification of metric-affine geometries
Why modified gravity? Basic geometrical quantities Tetrads and spin connection

Geometry and continuum mechanics

 In the context of continuum mechanics, the geometric tools of gravity have been used to describe various effects in continuum mechanics.

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- **Nonmetricity:** can be understood as crystalline structure with point defects (vacancies/intersticials)



Figure: Crystalline structure and its analogy with curvature, torsion and non-metricity



Figure: Crystal dislocations are shown against a regular crystal with no dislocation (left), and where *screw* (middle) and *edge* (right) dislocations are represented.

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Tetrads and the spin connection

• An alternative description of the metric-affine geometry outlined in the preceding section is to use a tetrad $e^a{}_{\mu}$ and a spin connection $\tilde{\omega}^a{}_{b\mu}$.

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Metric and tetrads

$$g_{\mu\nu} = \eta_{ab} e^a{}_{\mu} e^b{}_{\nu} , \quad g^{\mu\nu} = \eta^{ab} E_a{}^{\mu} E_b{}^{\nu}$$

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$$g_{\mu\nu} = \eta_{ab} e^a{}_{\mu} e^b{}_{\nu} , \quad g^{\mu\nu} = \eta^{ab} E_a{}^{\mu} E_b{}^{\nu}$$

where $\eta_{ab}={\rm diag}(1,-1,-1,-1)$ denotes the Minkowski metric.

Why modified gravity? Basic geometrical quantities Tetrads and spin connection

Tetrads fields

The vectors (\mathbf{E}_A) form an orthonormal basis of the tangent space, i.e.,

$$g(\mathbf{E}_{a}, \mathbf{E}_{b}) = g_{\mu\nu} E_{a}^{\ \mu} E_{b}^{\ \nu} = \eta_{ab} \,.$$
 (2)



Why modified gravity? Basic geometrical quantities Tetrads and spin connection

Spin connection and tetrads

• The frame coefficients $E_a{}^{\mu}$ are also required in order to calculate the coefficients $\tilde{\Gamma}^{\mu}{}_{\nu\rho}$ of the affine connection from the spin connection $\tilde{\omega}^a{}_{b\mu}$ via

$$\tilde{\Gamma}^{\rho}{}_{\mu\nu} = E_a{}^{\rho} \left(\partial_{\nu} e^a{}_{\mu} + \tilde{\omega}^a{}_{b\nu} e^b{}_{\mu} \right) \,, \tag{3}$$

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• This is the unique affine connection satisfying the so-called "tetrad postulate"

$$\partial_{\mu}e^{a}{}_{\nu}+\tilde{\omega}^{a}{}_{b\mu}e^{b}{}_{\nu}-\tilde{\Gamma}^{\rho}{}_{\nu\mu}e^{a}{}_{\rho}=0\,. \tag{4}$$

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Spin connection and tetrads

 One advantage of the formulation in terms of a tetrad and spin connection, is the fact that the curvature, torsion and non-metricity become properties of the spin connection only, and are independent of the choice of the tetrad.

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Spin connection and tetrads

- One advantage of the formulation in terms of a tetrad and spin connection, is the fact that the curvature, torsion and non-metricity become properties of the spin connection only, and are independent of the choice of the tetrad.
- Then, we can define the curvature, torsion and nonmetricity as:

$$\tilde{R}^{a}{}_{b\mu\nu} := \partial_{\mu}\tilde{\omega}^{a}{}_{b\nu} - \partial_{\nu}\tilde{\omega}^{a}{}_{b\mu} + \tilde{\omega}^{a}{}_{c\mu}\tilde{\omega}^{c}{}_{b\nu} - \tilde{\omega}^{a}{}_{c\nu}\tilde{\omega}^{c}{}_{b\mu} \,, \qquad (5)$$

$$\tilde{T}^{a}{}_{\mu\nu} := \partial_{\mu}e^{a}{}_{\nu} - \partial_{\nu}e^{a}{}_{\mu} + \tilde{\omega}^{a}{}_{b\mu}e^{b}{}_{\nu} - \tilde{\omega}^{a}{}_{b\nu}e^{b}{}_{\mu} \,, \tag{6}$$

$$\tilde{Q}_{\mu ab} := -\eta_{ac} \tilde{\omega}^c{}_{b\mu} - \eta_{cb} \tilde{\omega}^c{}_{a\mu} \,. \tag{7}$$

Trinity of gravity: GR, TEGR and STEGR.

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Trinity of gravity

As mentioned before, we can split the connection as

$$\tilde{\Gamma}^{\rho}{}_{\mu\nu} := \Gamma^{\rho}{}_{\mu\nu} + \tilde{K}^{\rho}{}_{\mu\nu} + \tilde{L}^{\rho}{}_{\mu\nu} := \Gamma^{\rho}{}_{\mu\nu} + \tilde{D}^{\rho}{}_{\mu\nu} \,, \tag{8}$$

where

$$\begin{split} \Gamma^{\mu}{}_{\nu\rho} &= \frac{1}{2} g^{\mu\sigma} \left(\partial_{\nu} g_{\sigma\rho} + \partial_{\rho} g_{\nu\sigma} - \partial_{\sigma} g_{\nu\rho} \right) , \quad \text{Levi Civita connection} \\ \tilde{K}^{\mu}{}_{\nu\rho} &= \frac{1}{2} \left(\tilde{T}_{\nu}{}^{\mu}{}_{\rho} + \tilde{T}_{\rho}{}^{\mu}{}_{\nu} - \tilde{T}^{\mu}{}_{\nu\rho} \right) , \quad \text{Contortion tensor} \\ \tilde{L}^{\mu}{}_{\nu\rho} &= \frac{1}{2} \left(\tilde{Q}^{\mu}{}_{\nu\rho} - \tilde{Q}_{\nu}{}^{\mu}{}_{\rho} - \tilde{Q}_{\rho}{}^{\mu}{}_{\nu} \right) , \quad \text{Disformation tensor} . \end{split}$$

frinity of gravity: GR, TEGR and STEGR.

Trinity of gravity - curvature tensor

• The curvature becomes

$$\tilde{R}^{\mu}{}_{\nu\rho\sigma} = R^{\mu}{}_{\nu\rho\sigma} + \overset{\circ}{\nabla}_{\rho}\tilde{D}^{\mu}{}_{\nu\sigma} - \overset{\circ}{\nabla}_{\sigma}\tilde{D}^{\mu}{}_{\nu\rho} + \tilde{D}^{\mu}{}_{\tau\rho}\tilde{D}^{\tau}{}_{\nu\sigma} - \tilde{D}^{\mu}{}_{\tau\sigma}\tilde{D}^{\tau}{}_{\nu\rho} \,.$$

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• Now, by contracting the curvature tensor to obtain the Ricci scalar $\tilde{R} = g^{\mu\nu} \tilde{R}^{\rho}{}_{\mu\rho\nu}$ we find

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Ricci scalar decomposition

$$\tilde{R} = R + \left(T + 2\nabla_{\mu}(\sqrt{-g}T^{\rho}{}_{\rho}{}^{\mu})\right) + \left(Q + \nabla_{\mu}Q^{\mu\nu}{}_{\nu} - \nabla_{\nu}Q_{\mu}{}^{\mu\nu}\right) + C$$

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with

$$\begin{split} T &:= T^{\rho\lambda\kappa}T_{\rho\lambda\kappa} + 2T^{\rho\lambda\kappa}T_{\kappa\rho\lambda} - 4T^{\kappa}_{\rho}T^{\rho\lambda}_{\kappa}T^{\rho\lambda}_{\lambda}, \quad \text{Torsion scalar}, \\ Q &:= -\frac{1}{4}Q_{\alpha\beta\gamma}Q^{\alpha\beta\gamma} + \frac{1}{2}Q_{\alpha\beta\gamma}Q^{\beta\alpha\gamma} + \frac{1}{4}Q_{\alpha}Q^{\alpha} - \frac{1}{2}Q_{\alpha}\bar{Q}^{\alpha}, \text{ Nonmetricity scalar} \\ C &:= 2(Q_{\kappa\rho\lambda}T^{\lambda\kappa\rho} + Q_{\rho}^{\ \sigma}{}_{\sigma}T^{\rho\kappa}{}_{\kappa} - Q^{\sigma}{}_{\sigma\rho}T^{\rho\kappa}{}_{\kappa}). \end{split}$$

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$$\tilde{R} = R + \left(T - 2\nabla_{\mu} (\sqrt{-g} T^{\rho}{}_{\rho}{}^{\mu})\right) + \left(Q + \nabla_{\mu} Q^{\mu\nu}{}_{\nu} - \nabla_{\nu} Q_{\mu}{}^{\mu\nu}\right) + \mathcal{L} = R.$$

Then, GR is constructed from the Ricci scalar

frinity of gravity: GR, TEGR and STEGR.

Trinity of gravity - General Relativity

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where $\kappa^2 = 8\pi G$ and $L_{\rm m}$ is any matter Lagrangian.

The Einstein's field equations are obtained by taking

variations w/r to the metric: $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} = \kappa^2 T_{\mu\nu}$.

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Sebastian Bahamonde

Black Holes in metric-affine

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Figure: Geometrical trinity of gravity (S. Bahamonde et.al., "Teleparallel Gravity: From Theory to Cosmology," [arXiv:2106.13793 [gr-qc]].)

<mark>Gauge formalis</mark>m Dynamics

MAG models with dynamical torsion and nonmetricity

Outline

- Trinity of gravity: GR, TEGR and STEGR. Metric-Affine gravity 3 Gauge formalism
 - Spherical symmetry
 - Observational constraints
 - Axial symmetry

<mark>Gauge formalism</mark> Dynamics

MAG models with dynamical torsion and nonmetricity

Gauge formalism of Poincaré gauge gravity

• Poincaré gauge gravity assumes zero nonmetricity $Q_{\alpha\mu\nu} = \nabla_{\alpha}g_{\mu\nu} = 0$ and a manifold with curvature and torsion.

Gauge formalism Dynamics

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Gauge formalism Dynamics

MAG models with dynamical torsion and nonmetricity

Gauge formalism of metric-affine geometry

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- Affine group A(4, R) = R⁴ ⊗ GL(4, R) is the semiproduct of the translation group R⁴ and the general linear group GL(4, R). gauge connection with an independent local metric structure¹:

$$A_{\mu} = e^{a}{}_{\mu}P_{a} + \omega^{a}{}_{b\mu}L_{a}{}^{b}, \qquad (9)$$

$$g_{\mu\nu} = e^a_{\ \mu} e^b_{\ \nu} \eta_{ab} \,. \tag{10}$$

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• Generators of the group $A(4, \mathcal{R})$:

$$[P_a, P_b] = 0, (11)$$

$$\left[L_a{}^b, P_c\right] = i\,\delta^b{}_c\,P_a\,,\tag{12}$$

$$[L_a{}^b, L_c{}^d] = i \left(\delta^b{}_c L_a{}^d - \delta_a{}^d L_c{}^b \right).$$
 (13)

Gauge formalism Dynamics

Gauge formalism of metric-affine geometry

• it is possible to obtain the following gauge curvatures from the anholonomic metric, coframe and connection:

$$G_{ab\mu} = \partial_{\mu}g_{ab} - g_{ac}\,\omega^c{}_{b\mu} - g_{bc}\,\omega^c{}_{a\mu}\,,\tag{14}$$

$$F^{a}_{\ \mu\nu} = \partial_{\mu}e^{a}_{\ \nu} - \partial_{\nu}e^{a}_{\ \mu} + \omega^{a}_{\ b\mu}e^{b}_{\ \nu} - \omega^{a}_{\ b\nu}e^{b}_{\ \mu} \,, \tag{15}$$

$$F^{a}{}_{b\mu\nu} = \partial_{\mu}\omega^{a}{}_{b\nu} - \partial_{\nu}\omega^{a}{}_{b\mu} + \omega^{a}{}_{c\mu}\omega^{c}{}_{b\nu} - \omega^{a}{}_{c\nu}\omega^{c}{}_{b\mu}.$$
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Correspondence with the curvature, torsion and nonmetricity tensors:

$$G_{ab\mu} = g_{ac}g_{bd}e^{c\lambda}e^{d\rho}Q_{\mu\lambda\rho}, \qquad (17)$$

$$F^{a}{}_{\mu\nu} = e^{a}{}_{\lambda}T^{\lambda}{}_{\nu\mu}, \qquad (18)$$

$$F^{a}{}_{b\mu\nu} = g_{bc} e^{a}{}_{\lambda} e^{c\rho} \tilde{R}^{\lambda}{}_{\rho\mu\nu} .$$
(19)

Gauge formalism Dynamics

MAG models with dynamical torsion and nonmetricity

Outline

- Trinity of gravity: GR, TEGR and STEGR. Metric-Affine gravity 3 Gauge formalism **Dynamics**
 - Spherical symmetry
 - Observational constraints
 - Axial symmetry

Gauge formalism Dynamics

MAG models with dynamical torsion and nonmetricity

Dynamics of metric-affine geometry

• Gravitational action with dynamical torsion and nonmetricity:

$$S = \int d^4x \sqrt{-g} \left[\mathcal{L}_m - \frac{1}{16\pi} \mathcal{L}_g(\tilde{\mathcal{R}}, \mathcal{T}, \mathcal{Q}) \right] \,.$$
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Gauge formalism Dynamics

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• Correspondence between geometry and matter:

$$\frac{\delta S_g}{\delta e^a{}_\nu} = 16\pi \theta_a{}^\nu, \qquad (21)$$
$$\frac{\delta S_g}{\delta \omega^a{}_{b\nu}} = 16\pi \Delta_a{}^{b\nu}. \qquad (22)$$

Here $\theta_a{}^{\nu}$ is the energy-momentum tensor (canonical) and $\Delta_a{}^{b\nu}$ is the hypermomentum density tensor.

Gauge formalism Dynamics

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Here $\theta_a{}^{\nu}$ is the energy-momentum tensor (canonical) and $\Delta_a{}^{b\nu}$ is the hypermomentum density tensor.

• *GL*(4, *R*) group allows the definition of a large number of scalar invariants depending on the aforementioned tensors.

Gauge formalism Dynamics

MAG models with dynamical torsion and nonmetricity

Dynamics of metric-affine geometry

• General quadratic gravitational action with dynamical torsion and nonmetricity:

$$S = \int d^{4}x \sqrt{-g} \left\{ \mathcal{L}_{m} + \frac{1}{16\pi} \left[-\tilde{R} + a_{1}\tilde{R}^{2} + a_{2}\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\lambda\rho\mu\nu} + a_{3}\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\rho\lambda\mu\nu} \right. \\ \left. + a_{4}\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\mu\nu\lambda\rho} + a_{5}\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\lambda\mu\rho\nu} + a_{6}\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\mu\lambda\rho\nu} + a_{7}\tilde{R}_{\rho\lambda\mu\nu}\tilde{R}^{\mu\lambda\rho\nu} \right. \\ \left. + a_{8}\tilde{R}_{\mu\nu}\tilde{R}^{\mu\nu} + a_{9}\tilde{R}_{\mu\nu}\tilde{R}^{\nu\mu} + a_{10}\hat{R}_{\mu\nu}\hat{R}^{\mu\nu} + a_{11}\hat{R}_{\mu\nu}\hat{R}^{\nu\mu} + a_{12}\tilde{R}_{\mu\nu}\hat{R}^{\mu\nu} \right. \\ \left. + a_{13}\tilde{R}_{\mu\nu}\hat{R}^{\nu\mu} + a_{14}\tilde{R}^{\lambda}_{\lambda\mu\nu}\tilde{R}^{\rho}_{\ \rho}^{\ \mu\nu} + a_{15}\tilde{R}^{\lambda}_{\lambda\mu\nu}\tilde{R}^{\mu\nu} + a_{16}\tilde{R}^{\lambda}_{\ \lambda\mu\nu}\hat{R}^{\mu\nu} \right. \\ \left. + b_{1}T_{\lambda\mu\nu}T^{\lambda\mu\nu} + b_{2}T_{\lambda\mu\nu}T^{\mu\lambda\nu} + b_{3}T^{\lambda}_{\lambda\nu}T^{\mu}_{\ \mu}^{\ \nu} + c_{1}T_{\lambda\mu\nu}Q^{\mu\lambda\nu} \right. \\ \left. + c_{2}T^{\lambda}_{\ \lambda\nu}Q^{\nu\mu}_{\ \mu} + c_{3}T^{\lambda}_{\ \lambda\nu}Q^{\mu\nu}_{\ \mu} + d_{1}Q_{\lambda\mu\nu}Q^{\lambda\mu\nu} + d_{2}Q_{\lambda\mu\nu}Q^{\mu\lambda\nu} \right. \\ \left. + d_{3}Q^{\lambda}_{\ \lambda\nu}Q^{\mu}_{\ \mu}^{\ \nu} + d_{4}Q_{\nu}^{\ \lambda}_{\ \lambda}Q^{\nu\mu}_{\ \mu} + d_{5}Q^{\lambda}_{\ \lambda\nu}Q^{\nu\mu}_{\ \mu} \right] \right\}.$$
 (23)

Spherical symmetry Observational constraints Axial symmetry

MAG models with dynamical torsion and nonmetricity

• In order to have a theory such that when T = Q = 0 one recovers GR, one can relate the constants.

²S. Bahamonde and J. G. Valcarcel, JCAP **09**, 057 (2020).

³S. Bahamonde and J. G. Valcarcel, JCAP **01** (2022) no.01, 011.

MAG models with dynamical torsion and nonmetricity

- In order to have a theory such that when T = Q = 0 one recovers GR, one can relate the constants.
- Quadratic gravitational action with dynamical torsion and nonmetricity in Weyl-Cartan geometry ($Q_{\lambda\mu\nu} = g_{\mu\nu}W_{\lambda}$)

$$S = \int d^{4}x \sqrt{-g} \Big\{ \mathcal{L}_{m} + \frac{1}{64\pi} \Big[-4R - 6d_{1}\tilde{R}_{\lambda[\rho\mu\nu]}\tilde{R}^{\lambda[\rho\mu\nu]} \\ -9d_{1}\tilde{R}_{\lambda[\rho\mu\nu]}\tilde{R}^{\mu[\lambda\nu\rho]} + 8d_{1}\tilde{R}_{[\mu\nu]}\tilde{R}^{[\mu\nu]} + \frac{1}{8} \left(32e_{1} + 8e_{2} + 17d_{1} \right) \tilde{R}^{\lambda}{}_{\lambda\mu\nu}\tilde{R}^{\rho}{}_{\rho}{}^{\mu\nu} \\ -7d_{1}\tilde{R}_{[\mu\nu]}\tilde{R}^{\lambda}{}_{\lambda}{}^{\mu\nu} + 3\left(1 - 2a_{2} \right) T_{[\lambda\mu\nu]}T^{[\lambda\mu\nu]} \Big] \Big\}.$$

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 Absence of a general Birkhoff's theorem in MAG: new spherically and axially symmetric vacuum solutions with independent dynamical torsion and nonmetricity fields^{2,3}

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Spherical symmetry Observational constraints Axial symmetry

Outline

- Trinity of gravity: GR, TEGR and STEGR. Gauge formalism MAG models with dynamical torsion and nonmetricity Spherical symmetry
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Spherical symmetry Observational constraints Axial symmetry

Spherical symmetry

• Metric, torsion and nonmetricity in spherically symmetric space-times (#2 + #8 + #2 = #12):

$$\mathcal{L}_{\xi}g_{\mu\nu} = \mathcal{L}_{\xi}T^{\lambda}{}_{\mu\nu} = \mathcal{L}_{\xi}W_{\mu} = 0 \implies \mathcal{L}_{\xi}\tilde{R}_{\lambda\rho\mu\nu} = 0$$

Spherical symmetry Observational constraints Axial symmetry

Spherical symmetry

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$$\mathcal{L}_{\xi}g_{\mu\nu} = \mathcal{L}_{\xi}T^{\lambda}{}_{\mu\nu} = \mathcal{L}_{\xi}W_{\mu} = 0 \implies \mathcal{L}_{\xi}\tilde{R}_{\lambda\rho\mu\nu} = 0$$

 By solving these equations we find that torsion and nonmetricity behave as

$$T^{t}{}_{tr} = a(r), \quad T^{r}{}_{tr} = b(r), \quad T^{\theta_{k}}{}_{t\theta_{k}} = f(r), \quad T^{\theta_{k}}{}_{r\theta_{k}} = g(r)$$

$$T^{\theta_{k}}{}_{t\theta_{l}} = e^{a\theta_{k}} e^{b}{}_{\theta_{l}} \epsilon_{ab} d(r), \quad T^{\theta_{k}}{}_{r\theta_{l}} = e^{a\theta_{k}} e^{b}{}_{\theta_{l}} \epsilon_{ab} h(r),$$

$$T^{t}{}_{\theta_{k}\theta_{l}} = \epsilon_{kl} k(r) \sin \theta_{1}, \quad T^{r}{}_{\theta_{k}\theta_{l}} = \epsilon_{kl} l(r) \sin \theta_{1},$$

$$W_{\lambda} = (w_{1}(r), w_{2}(r), 0, 0),$$

whereas the metric is in the standard spherically symmetric form:

$$ds^{2} = \Psi_{1}(r) dt^{2} - \frac{dr^{2}}{\Psi_{2}(r)} - r^{2} \left(d\theta_{1}^{2} + \sin^{2} \theta_{1} d\theta_{2}^{2} \right) \,.$$

Here, ϵ_{kl} is the Levi-Civita symbol in two dimensions.

Spherical symmetry Observational constraints Axial symmetry

Spherical symmetry - Solving the field equations

The field eqs are very involved. To solve them we use the following strategy:

• Imposing regularity: In general, the solutions can have a singular behaviour. To ensure regularity, one can analyse the torsion/nonmetricity tensors referred to the rotated basis $\vartheta^a = \Lambda^a{}_b e^b$.

Spherical symmetry Observational constraints Axial symmetry

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Spherical symmetry - Solving the field equations

The field eqs are very involved. To solve them we use the following strategy:

• Imposing regularity: In general, the solutions can have a singular behaviour. To ensure regularity, one can analyse the torsion/nonmetricity tensors referred to the rotated basis $\vartheta^a = \Lambda^a{}_b e^b$.

One can write the gauge curvature $\mathcal{F}^a{}_{bc} = \vartheta^a{}_\lambda \vartheta_b{}^\mu \vartheta_c{}^\nu T^\lambda{}_{\nu\mu}$ related to the torsion/nonmetricity tensor in this orthogonal coframe.

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Regularity restricts the initial arbitrariness of the torsion components and the Weyl vector by imposing the relations

$$b(r) = a(r) \sqrt{\Psi_1(r)\Psi_2(r)}, \qquad f(r) = -g(r) \sqrt{\Psi_1(r)\Psi_2(r)},$$

$$d(r) = -h(r) \sqrt{\Psi_1(r)\Psi_2(r)}, \qquad l(r) = k(r) \sqrt{\Psi_1(r)\Psi_2(r)},$$

$$w_1(r) = -w_2(r) \sqrt{\Psi_1(r)\Psi_2(r)}.$$

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Spherical symmetry - Solving the field equations

Oslve the weak field limit: The weak field limit of the field equations become

$$\nabla_{\rho} \nabla_{\lambda} T^{\lambda \rho}{}_{\mu} + \nabla_{\rho} \nabla^{\rho} T^{\lambda}{}_{\mu \lambda} - \nabla_{\rho} \nabla_{\mu} T^{\lambda \rho}{}_{\lambda} = 0,$$

$$\nabla_{\mu} \tilde{R}^{\lambda}{}_{\lambda}{}^{\mu \nu} = 0.$$

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$$\nabla_{\mu} \tilde{R}^{\lambda}{}_{\lambda}{}^{\mu \nu} = 0.$$

These equations can be solved, yielding

$$w_1(r) = -\kappa_d \int \sqrt{\frac{\Psi_1(r)}{\Psi_2(r)}} \frac{dr}{r^2},$$

$$b(r) = rf'(r) + f(r) + \frac{\kappa_d}{2r} \sqrt{\frac{\Psi_1(r)}{\Psi_2(r)}},$$

where κ_d is an integration constant which represents the dilaton charge.

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Spherical symmetry - Solving the field equations

The final solution for the metric behaves as Reissner-Nordström

$$g_{tt} = -1/g_{rr} \equiv \Psi(r) = 1 - \frac{2m}{r} + \frac{d_1\kappa_s^2 - 4e_1\kappa_{d,e}^2}{r^2}$$
. (24)

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Ontermosticity sector:

$$W_{\mu} = \frac{\kappa_{\rm d,e}}{r} \left(1, -1/\Psi(r), 0, 0 \right) \,. \tag{25}$$

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Torsion sector:

$$\bar{\mathcal{S}}^{a} = -\frac{\kappa_{\rm s}}{r} (1, 1, 0, 0), \qquad (26)$$

$$\bar{\mathcal{T}}_{2}^{abc} = \frac{\kappa_{\rm s}}{3r} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}. \qquad (27)$$

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Dilation and spin charges

What do κ_s (dilation charge) and $\kappa_{d,e}$ (spin charge) physically represent?

Point 1 - Hypermomentum density

In the geometric scheme of MAG, not only an energy-momentum tensor of matter arises as source of curvature, but also a hypermomentum density tensor which operates as source of torsion and nonmetricity.

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Point 2 - Dilation and spin charges

In Weyl-Cartan geometry, hypermomentum density tensor splits into spin and dilation currents, which carry their own charges and provide a RN solution.

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Dilation and spin charges

When these charges might be important?

Significant effects are contemplated only around **extreme gravitational systems**, such as **neutron stars** with intense magnetic fields and sufficiently oriented elementary spins or **black holes** endowed with spin and dilation charges.

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Outline

- Trinity of gravity: GR, TEGR and STEGR. Gauge formalism MAG models with dynamical torsion and nonmetricity Spherical symmetry
 - Observational constraints
 - Axial symmetry

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Nature of the black hole solution

• The nature of the horizons depends on the difference $d_1\kappa_s^2 - 4e_1\kappa_d^2$. Thus, a positive difference of this quantity would present two horizons determined from the roots

$$r_{\pm} = M \pm \Delta_1, \quad \Delta_1^2 = M^2 - \left(d_1 \kappa_s^2 - 4 e_1 \kappa_d^2 \right),$$

with $0 < \left(d_1\kappa_s^2 - 4e_1\kappa_d^2\right) < M^2$.

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• The different signs of the kinetic terms related to the dynamical part of torsion and to the Weyl vector allows the case $d_1\kappa_s^2 - 4e_1\kappa_d^2 < 0$.

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- The different signs of the kinetic terms related to the dynamical part of torsion and to the Weyl vector allows the case $d_1\kappa_s^2 4e_1\kappa_d^2 < 0$.
- The balance between κ_d and κ_s is not restricted to any special constraint and therefore any of these situations may occur in the presence of torsion and nonmetricity.

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Particle motion in MAG

 The equations of motion of test bodies with microstructure coupled to the torsion and nonmetricity tensors become⁴

$$\dot{p}^{\mu} + \Gamma^{\mu}{}_{\lambda\rho} p^{\lambda} u^{\rho} + N_{[\lambda\rho]}{}^{\mu} p^{\rho} u^{\lambda} + \tilde{R}_{\lambda\rho\sigma}{}^{\mu} \bigtriangleup^{\rho\lambda} u^{\sigma} = 0.$$

⁴D. Puetzfeld and Y. N. Obukhov, Phys. Rev. D **76** (2007), 084025.

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• This eq. reduces to the standard geodesic one $(\dot{p}^{\mu} + \Gamma^{\mu}_{\lambda\rho} p^{\lambda} u^{\rho} = 0)$ when the hypermomentum of the test body vanishes and also when the particle are bosons.

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- This eq. reduces to the standard geodesic one $(\dot{p}^{\mu} + \Gamma^{\mu}_{\lambda\rho} p^{\lambda} u^{\rho} = 0)$ when the hypermomentum of the test body vanishes and also when the particle are bosons.
- Using the standard geodesic approach, we find

$$\frac{1}{2}\dot{r}^2 + V(r) = 0\,, \quad V(r) = -\frac{1}{2}c^2E^2 + \frac{1}{2}\Psi(r)\left(\frac{J^2}{r^2} + \sigma c^2\right)\,,$$

where *E* an *J* are the conserved charges and $\sigma = 0(\sigma = 1)$ represents massless(massive) particles.

⁴D. Puetzfeld and Y. N. Obukhov, Phys. Rev. D **76** (2007), 084025.

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Photon sphere and perihelion shift

• **Photon sphere:** Region of space where gravity is so strong that photons are forced to travel in orbits. They are found by setting $V'(r) = V(r) = \sigma = 0$, giving us:

$$r_{1} = \frac{G}{2c^{2}} \left(3M + \Delta_{2} \right), \quad J_{1,\pm} = \pm \frac{GE(\Delta_{2} + 3M)^{2}}{\sqrt{2}c\sqrt{\Delta_{2}^{2} + 3M^{2} + 4\Delta_{2}M}},$$

$$r_{2} = \frac{G}{2c^{2}} \left(3M - \Delta_{2} \right), \quad J_{2,\pm} = \pm \frac{GE(\Delta_{2} - 3M)^{2}}{\sqrt{2}c\sqrt{\Delta_{2}^{2} + 3M^{2} - 4\Delta_{2}M}},$$

where we have defined $\Delta_2^2 := M^2 + 8\Delta_1^2 = 9M^2 - \frac{8}{G} \left(d_1 \kappa_s^2 - 4e_1 \kappa_d^2 \right).$

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 The first pair (r₁, J_{1,±}) describes a unique photon sphere that lies outside the event horizons, with the corrections related to κ_s, κ_d affecting its location with respect to the Schwarzschild solution of GR. Introduction to Metric-affine gravity Trinity of gravity Metric-Affine gravity

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MAG models with dynamical torsion and nonmetricity

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• **Perihelion shift:** We consider a massive body with $\sigma = 1$ and a perturbation around its closed orbit r_c $(\dot{r}_c = V(r_c) = V'(r_c) = 0)$:

$$\begin{split} \Delta\phi &= 2\pi \left[\frac{3GM}{c^2 r_c} + \frac{27G^2M^2}{2c^4 r_c^2} + \frac{135G^3M^3}{2c^6 r_c^3} + \frac{2835G^4M^4}{8c^8 r_c^4} \right. \\ &- d_1 \kappa_s^2 \left(\frac{1}{2c^2 M r_c} + \frac{6G}{c^4 r_c^2} \right) + e_1 \kappa_d^2 \left(\frac{2}{Mc^2 r_c} + \frac{24G}{c^4 r_c^2} \right) \right] \end{split}$$

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 One could expect that these contributions coming from metric-affine geometry will be only sourced in a strong gravitational regime, e.g., Sgr A* and S2 stars.

⁵B. Abuter et al. [GBAVITY]. Astron. Astrophys. 636 (2020). L5

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 One could expect that these contributions coming from metric-affine geometry will be only sourced in a strong gravitational regime, e.g., Sgr A* and S2 stars.

• Since
$$\Delta \phi_{S_2}^{(GR)} \approx 48.550 \, [''/year]$$
 and
 $\Delta \phi_{S_2}^{(obs)} = 48.506 f_{SP} \, [''/year]^5$ we find the constrain
 $4e_1 \kappa_d^2 - 5.711 \cdot 10^{63} \, [J \cdot m] \le d_1 \kappa_s^2 \le 4e_1 \kappa_d^2 + 2.894 \cdot 10^{63} \, [J \cdot m]$

⁵ B. Abuter et al. [GBAVITY]. Astron. Astrophys. 636 (2020). L5

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Gravitational redshift

 Another gravitational effect that can be used to constrain the new effects arising from our model is the gravitational redshift, that for our solution we get

$$z = \frac{GM}{c^2R} + \frac{3G^2M^2}{2c^4R^2} + \frac{5G^3M^3}{2c^6R^3} + \frac{35G^4M^4}{8c^8R^4} + \frac{G\left(4e_1\kappa_d^2 - d_1\kappa_s^2\right)}{2c^4R^2}$$

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- We can now consider astrophysical compact objects, such as degenerate stars composed by fermionic matter with a spin alignment induced by torsion.
- Current measurements for the masses and gravitational redshifts of isolated neutron stars do not provide independent quantities.

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- We can now consider astrophysical compact objects, such as degenerate stars composed by fermionic matter with a spin alignment induced by torsion.
- Current measurements for the masses and gravitational redshifts of isolated neutron stars do not provide independent quantities.
- We focused on the Sirius B white dwarf which gives us

 $4e_1\kappa_d^2 - 2.931\cdot 10^{43}~[J\cdot m] \le d_1\kappa_s^2 \le 4e_1\kappa_d^2 + 1.016\cdot 10^{43}~[J\cdot m]~.$

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Observational constrains

• Let us now consider the case where the effect of torsion dominates over the contribution of nonmetricity.

⁶S. Bahamonde and J. Gigante Valcarcel, Eur. Phys. J. C 81 (2021) no.6, 495.

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- Let us now consider the case where the effect of torsion dominates over the contribution of nonmetricity.
- Indeed, due to the presence of a magnetic field in white dwarfs, it is expected that Sirius B can have sufficiently oriented elementary spins in comparison with an effective dilation charge, therefore, $\kappa_{s,SiriusB} \gg \kappa_{d,SiriusB}$.

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- Indeed, due to the presence of a magnetic field in white dwarfs, it is expected that Sirius B can have sufficiently oriented elementary spins in comparison with an effective dilation charge, therefore, $\kappa_{s,SiriusB} \gg \kappa_{d,SiriusB}$.
- **Perihelion shift+ Gravitational redshift:** Assuming the same approximation in Sgr A* and considering the universality of the coupling constant *d*₁, we find⁶

$$1.396 \cdot 10^{10} \le \frac{\kappa_{s, \text{SgrA}*}}{\kappa_{s, \text{SiriusB}}} \le 1.688 \cdot 10^{10}$$

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- **Perihelion shift+ Gravitational redshift:** Assuming the same approximation in Sgr A* and considering the universality of the coupling constant *d*₁, we find⁶

 $1.396 \cdot 10^{10} \le \frac{\kappa_{s, \text{SgrA}*}}{\kappa_{s, \text{SiriusB}}} \le 1.688 \cdot 10^{10} \,.$

 To the best of our knowledge, this bound provides the first observational comparison between the spin charges of a <u>supermassive black hole</u> and a degenerate star.

⁶S. Bahamonde and J. Gigante Valcarcel, Eur. Phys. J. C 81 (2021) no.6, 495.

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Outline

- Trinity of gravity: GR, TEGR and STEGR. Gauge formalism MAG models with dynamical torsion and nonmetricity Spherical symmetry
 - Observational constraint
 - Axial symmetry

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Extension to axisymmetric space-times

 Metric, torsion and nonmetricity tensors in symmetric space-times:

$$\mathcal{L}_{\xi}g_{\mu\nu} = \mathcal{L}_{\xi}T^{\lambda}{}_{\mu\nu} = \mathcal{L}_{\xi}Q^{\lambda}{}_{\mu\nu} = 0 \implies \mathcal{L}_{\xi}\tilde{R}^{\lambda}{}_{\rho\mu\nu} = 0.$$
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 (28)

• Stationary and axisymmetric space-times:

$$#10 \to #4 \begin{cases} ds^2 = \Psi_1(r,\vartheta) dt^2 - \frac{dr^2}{\Psi_2(r,\vartheta)} \\ -r^2 \Psi_3(r,\vartheta) \Big[d\vartheta^2 + \sin^2 \vartheta (d\varphi - \Psi_4(r,\vartheta) dt)^2 \Big] \end{cases};$$

$$#24\left\{T^{\lambda}_{\mu\nu} = T^{\lambda}_{\mu\nu}(r,\vartheta)\right\}$$
(29)

$$#4\left\{W_{\mu} = (W_t(r,\vartheta), W_r(r,\vartheta), W_{\vartheta}(r,\vartheta), W_{\varphi}(r,\vartheta)).\right.$$
(30)

Axisymmetric space-times - Kerr-Newmann de-Sitter

• Rotating Kerr-Newman metric structure⁷:

$$ls^{2} = \Psi(r,\vartheta) dt^{2} - \frac{r^{2} + a^{2} \cos^{2} \vartheta}{(r^{2} + a^{2} \cos^{2} \vartheta) \Psi(r,\vartheta) + a^{2} \sin^{2} \vartheta} dr^{2}$$
$$- \left(r^{2} + a^{2} \cos^{2} \vartheta\right) d\vartheta^{2} + 2a \left(1 - \Psi(r,\vartheta)\right) \sin^{2} \vartheta dt d\varphi$$
$$- \sin^{2} \vartheta \left[r^{2} + a^{2} + a^{2} \left(1 - \Psi(r,\vartheta)\right) \sin^{2} \vartheta\right] d\varphi^{2},$$
(31)

$$\Psi(r,\vartheta) = 1 - \frac{\left[2mr + 4e_1(\kappa_{d,e}^2 + \kappa_{d,m}^2) - d_1\kappa_s^2\right]}{r^2 + a^2\cos^2\vartheta} \,. \tag{32}$$

⁷S. Bahamonde and J. G. Valcarcel. JCAP 01 (2022) no.01.011.

Axisymmetric space-times - Kerr-Newmann de-Sitter

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$$- \left(r^{2} + a^{2} \cos^{2} \vartheta\right) d\vartheta^{2} + 2a \left(1 - \Psi(r,\vartheta)\right) \sin^{2} \vartheta dt d\varphi$$
$$- \sin^{2} \vartheta \left[r^{2} + a^{2} + a^{2} \left(1 - \Psi(r,\vartheta)\right) \sin^{2} \vartheta\right] d\varphi^{2},$$
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 (32)

Field strength tensors:

$$\bar{R}_{[\mu\nu]} = \frac{1}{12} \varepsilon^{\lambda} \sigma_{\mu\nu} \nabla_{\lambda} \bar{S}^{\sigma} + \frac{1}{2} \nabla_{\lambda} \bar{t}^{\lambda} {}_{\mu\nu}; \quad \tilde{R}^{\lambda} {}_{\lambda\mu\nu} = 4 \nabla_{[\nu} W_{\mu]};$$

$$\bar{R}^{\lambda} {}_{[\mu\nu\rho]} = \frac{1}{6} \varepsilon^{\lambda} {}_{\sigma[\rho\nu} \nabla_{\mu]} \bar{S}^{\sigma} + \nabla_{[\mu} \bar{t}^{\lambda} {}_{\rho\nu]} + \frac{1}{4} \varepsilon^{\lambda} {}_{\omega\sigma[\rho} \hat{t}^{\sigma}_{1} {}_{\mu\nu]} \bar{S}^{\omega} - \frac{1}{18} \varepsilon_{\sigma\mu\nu\rho} \hat{T}^{\lambda}_{1} \bar{S}^{\sigma}.$$
(33)

⁷S. Bahamonde and J. G. Valcarcel, JCAP 01 (2022) no.01, 011.

Axisymmetric space-times - Kerr-Newmann de-Sitter

Nonmetricity sector:(no approx.)

$$w_{1}(r,\vartheta) = \frac{\kappa_{\mathrm{d,e}}r - a \kappa_{\mathrm{d,m}} \cos\vartheta}{r^{2} + a^{2} \cos^{2}\vartheta}, \quad w_{3}(r,\vartheta) = 0,$$

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Axisymmetric space-times - Kerr-Newmann de-Sitter

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• Torsion sector (decoupling limit between the spin and the orbital angular momentum $|a\kappa_{\rm s}|\ll 1$):

$$\bar{\mathcal{S}}^{a} = -\frac{\kappa_{\rm s}}{r} (1, 1, 0, 0) + \mathcal{O}(a\kappa_{\rm s}),$$

$$\bar{\mathcal{T}}_{2}^{abc} = \frac{\kappa_{\rm s}}{3r} \begin{pmatrix} 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 \end{pmatrix} + \mathcal{O}(a\kappa_{\rm s}).$$
(36)

Spherical symmetry Observational constraints Axial symmetry

Gravitational spin-orbit interaction

• We found a solution in the decoupling limit $a\kappa_s \ll 1$, which ensures that the Maxwell equation and closure conditions are fulfilled by the field strength tensors of torsion

$$\nabla_{\lambda} \tilde{R}^{\lambda}{}_{[\rho\mu\nu]} = \nabla_{\mu} \tilde{R}^{[\mu\nu]} = 0 \,, \quad \nabla_{[\sigma} \tilde{R}_{\lambda[\rho\mu\nu]]} = \nabla_{[\lambda} \tilde{R}_{[\mu\nu]]} = 0 \,.$$

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Gravitational spin-orbit interaction:

$$\mathcal{H}_{\rm LS} = \frac{1}{m_{\rm e}^2 r} \frac{\partial V}{\partial r} \mathbf{L} \cdot \mathbf{S} \approx \frac{d_1}{2r} \frac{\partial g_{tt}}{\partial r} a \kappa_{\rm s} \cos \vartheta$$
(37)

Extension to axisymmetric space-times - Plebanski-Damianski

 It is well known that the most general axisymmetric system in vacuum that can describe a BH type D in GR contains⁸:

Mass	M
Angular momentum	a
Taub-NUT charge	l
Acceleration	α

⁸J. F. Plebanski and M. Demianski, Annals Phys. **98** (1976), 98-127

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- The solution in GR is called Plebanski-Damianski solution.

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• The Plebanski-Damianski metric was recently presented in an improved form with $\Lambda = 0$ in by Podolský and Vrátný (Phys. Rev. D **104** (2021), 084078), and it can be written as

$$ds^{2} = \Omega^{-2}(r,\vartheta) \left\{ \Phi_{1}(r,\vartheta) \left[dt - \left(a \sin^{2}\vartheta + 2l(\chi - \cos\vartheta) \right) d\varphi \right]^{2} - \frac{dr^{2}}{\Phi_{1}(r,\vartheta)} - \frac{d\vartheta^{2}}{\Phi_{2}(r,\vartheta)} - \Phi_{2}(r,\vartheta) \sin^{2}\vartheta \left[a dt - \left(r^{2} + a^{2} + l^{2} + 2\chi al \right) d\varphi \right]^{2} \right\}.$$

where Φ_i, Ω are cumbersome functions depending on these parameters.

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where Φ_i, Ω are cumbersome functions depending on these parameters.

• We just found this new form with the cosmological constant⁹ with $\Phi_1(r, \vartheta) = \frac{Q(r)}{\rho^2(r, \vartheta)}$, $\Phi_2(r, \vartheta) = \frac{P(\vartheta)}{\rho^2(r, \vartheta)}$, and $\rho^2(r, \vartheta) = r^2 + (a \cos \vartheta + l)^2$. Here, $Q(r), \Omega(\vartheta)$ include the PD <u>quantities.</u>

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• We found a solution to **OUR THEORY** in the decoupling limit $|x_i\kappa_s| \ll 1$ with $x = (a, l, \alpha)$ with additional torsion and nonmetricity terms

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Spherical symmetry Observational constraints Axial symmetry

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$$\begin{split} w_1(r,\vartheta) &= \frac{\kappa_{\mathrm{d},\mathrm{e}}r - \kappa_{\mathrm{d},\mathrm{m}}(a\cos\vartheta + l)}{r^2 + (a\cos\vartheta + l)^2}, \quad w_2(r,\vartheta) = -\frac{\kappa_{\mathrm{d},\mathrm{e}}r - \kappa_{\mathrm{d},\mathrm{m}}(a\gamma + l)}{Q(r)}, \\ w_3(r,\vartheta) &= -\kappa_{\mathrm{d},\mathrm{m}}\sqrt{K(\vartheta) - \left(\frac{\cot\vartheta - \gamma\csc\vartheta}{P(\vartheta)}\right)^2}, \\ w_4(r,\vartheta) &= \kappa_{\mathrm{d},\mathrm{m}} \left[\frac{\left(r^2 + a^2 - l^2\right)\cos\vartheta + al\sin^2\vartheta + 2\chi l\left(a\cos\vartheta + l\right)}{r^2 + (a\cos\vartheta + l)^2} - \gamma\right] \\ &- \frac{\kappa_{\mathrm{d},\mathrm{e}}r\left[a\sin^2\vartheta + 2l\left(\chi - \cos\vartheta\right)\right]}{r^2 + (a\cos\vartheta + l)^2}, \\ \bar{T}^\vartheta \varphi_t &= -\bar{T}^\varphi_{\ \vartheta t}\sin^2\vartheta = -\bar{T}^\vartheta \varphi_r \frac{Q(r)}{\rho^2(r,\vartheta)} = \bar{T}^\varphi_{\ \vartheta r} \frac{Q(r)}{\rho^2(r,\vartheta)}\sin^2\vartheta = \frac{\kappa_s\sin\vartheta}{r} + \mathcal{O}(x_i\kappa_s) \end{split}$$

 Similarly as electromagnetism, the torsion behaves as a Coulomb-like quantity depending on a spin charge κ_s and the non-metricity on the dilation charge κ_d.

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Conclusions - Messages to home

 GR assumes different assumptions and there are some good indications that GR should not be the final theory of gravity
modified gravity?

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- There are three alternative ways of representing gravity as GR and they are indistinguishable at the classical level (GR, TEGR, STEGR).
- The MAG are gauge theories of gravity with the field strength tensors given by the curvature, torsion and nonmetricity.

Spherical symmetry Dbservational constraints Axial symmetry

Conclusions

 In the 1st paper we found an exact black hole solution in a MAG theory with torsion and nonmetricity being dynamical and independent.

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- In a 4th paper we found Plebanski-Demianski uniformly accelerated rotating black hole solutions with NUT parameter, electromagnetic charges and a Λ .
- Future: search of a gravitational spin-orbit interaction in MAG beyond the Kerr-Newman space-time (MAG is the main candidate to describe a spin-orbit interaction beyond GR).