Teleparallel theories of gravity and applications to cosmology

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Outline

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1. Introduction to Teleparallel theories of gravity
   - Basic mathematical ingredients
   - Teleparallel equivalent of General Relativity
2. Modified Teleparallel theories of gravity
   - General features
   - Some important theories
3. Applications to cosmology
   - How to study cosmology in Teleparallel gravity
   - Summary of results
4. Conclusions and final remarks

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where η_{ab} is the Minkowski metric.

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- In TG, it is always possible to find a frame such that $\omega^a{}_{b\mu} = 0$, but this is a gauge choice, so only some tetrads are compatible with this.

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- The pure tetrad formalism was the initial framework used for TG, which chooses a specific frame where the spin connection $\omega^a{}_{b\mu}$ vanishes. Be careful choosing the correct tetrad which is compatible with this gauge.

Local Lorentz transformations

- If one performs a local Lorentz transformation $e'^{a}{}_{\mu} = \Lambda^{a}{}_{b} e^{b}{}_{\mu}$, the metric $g_{\mu\nu} = g'_{\mu\nu}$ is invariant. A consequence of this is that the metric has 10 d.o.f. and the tetrads 10+6(the extra comes from $\Lambda^{a}{}_{b}$). Different tetrads can give the same metric.

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- The torsion tensor is covariant under local Lorentz transformation.

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Curvature in Teleparallel gravity

$$R^{a}{}_{b\mu\nu}(\omega^{a}{}_{b\mu}) = \partial_{\mu}\omega^{a}{}_{b\nu} - \partial_{\nu}\omega^{a}{}_{b\mu} + \omega^{a}{}_{c\mu}\omega^{c}{}_{b\nu} - \omega^{a}{}_{c\nu}\omega^{c}{}_{b\mu} \equiv 0\,.$$

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- **Be careful here!** The general curvature $R^{\lambda}{}_{\mu\sigma\nu}\equiv 0$, not $\mathring{R}^{\lambda}{}_{\mu\sigma\nu}\neq 0$

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Equivalence between field equations

The field equations arising from $S_{\rm TEGR}$ are equivalent to the Einstein field equations.

Two different ways of understanding gravity

Coupling to matter

In TG, no direct matter coupling to the teleparallel connection is introduced, in order to preserve the weak equivalence principle $\Longrightarrow$ matter fields retain their universal **coupling to the metric** and possibly its **Levi-Civita connection** (in the case of spinor fields).

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Equivalence on their field equations

TEGR has the same equations as GR, so **CLASSICALLY** it is impossible to make any observation to distinguish between them. All **classical experiments** already done, that have confirmed GR, also can be understood as a confirmation of TEGR.

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$\overset{\diamond}{Q}_\alpha = \overset{\diamond}{Q}_{\alpha\lambda}{}^\lambda$ and $\tilde{\overset{\diamond}{Q}}_\alpha = \overset{\diamond}{Q}{}^\lambda{}_{\lambda\alpha}$.

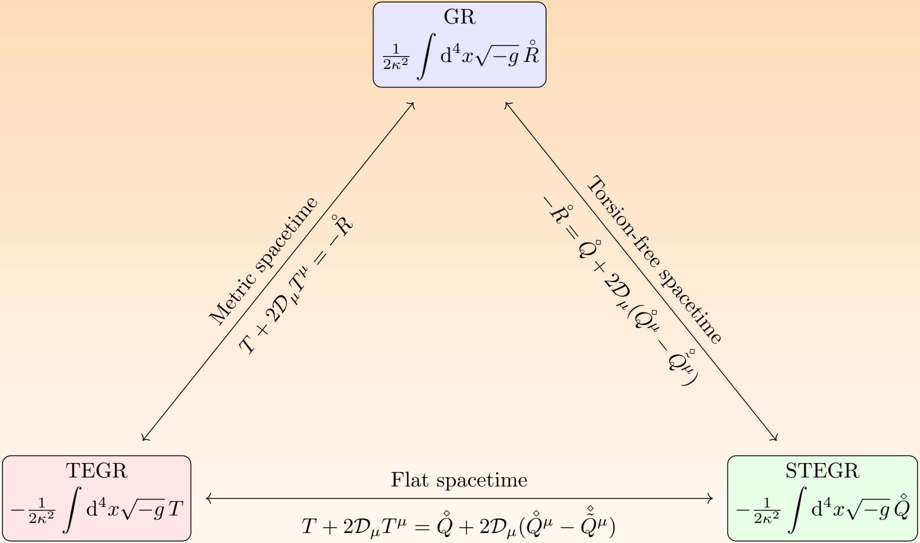

Figure: Geometrical trinity of gravity

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Modified teleparallel theories

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Are Teleparallel theories Lorentz invariant?

There is a lot of misconceptions in the literature since TEGR and their first modifications were proposed in the pure-tetrad formalism, which assumes that ${\omega^a}_{b\mu} = 0$ globally. By assuming this choice, TEGR is pseudo local Lorentz invariant (invariant up to a boundary term) and in modified TG, there is a breaking of the local Lorentz invariant.

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- **Gauge nature:** Teleparallel gravity can be written as the gauge theory of translations with the torsion tensor being the field strength of the theory.
- **The division between inertial and gravitational effects:** In GR, gravity and inertia cannot be separated, but in TG, there are some arguments in favour that it is possible to separate them by putting all the inertial effects in ${\omega^a}_{b\mu}$ and gravity in the tetrads ${e^a}_\mu$.

Important properties of Teleparallel theories

- Teleparallel theories have the tetrads and spin connection as the fundamental variables, so that, one most commonly assumes an action which is of the form

$$\mathcal{S} = \mathcal{S}_{\mathsf{g}}[e,\omega] + \mathcal{S}_{\mathsf{m}}[e,\chi]\,,$$

where the gravitational part $\mathcal{S}_{\mathsf{g}}$ of the action depends on the tetrad $e^{A}{}_{\mu}$ and the spin connection $\omega^{A}{}_{B\mu}$, while the matter part depends on the tetrad $e^{A}{}_{\mu}$ and arbitrary matter fields χ^{I}, but not on the spin connection.

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- Particles (bosonic or femionic) follow the standard geodesic equation.

Important properties of Teleparallel theories

- Variations of any action should be taken with respect to both variables (with the emphasis that the spin connection is non-arbitrary but always flat).

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- Variations of any action should be taken with respect to both variables (with the emphasis that the spin connection is non-arbitrary but always flat).
- Since ${\omega^A}_{B\mu}$ is a pure-gauge quantity, it can be shown that the the antisymmetric part of the field equations arising from variations w/r to the tetrads ${e^A}_\mu$ coincides with the variations of the action w/r to ${\omega^A}_{B\mu}$.

Important properties of Teleparallel theories

- Field equations are fully covariant (Lorentz and diffeo), and after finding them, it is possible to choose a gauge, known as the Weitzenböck gauge which is a special frame where $\omega^{A}{}_{B\mu} = 0$.

Important properties of Teleparallel theories

- Field equations are fully covariant (Lorentz and diffeo), and after finding them, it is possible to choose a gauge, known as the Weitzenböck gauge which is a special frame where ${\omega^A}_{B\mu} = 0$.
- This gauge choice can be only taken after deriving the field equations and if one does this, only some tetrads will be compatible with this choice.

New General Relativity (NGR)

The torsion tensor can be decomposed in its irreducible parts as

$$a_\mu = \frac{1}{6}\epsilon_{\mu\nu\sigma\rho}T^{\nu\sigma\rho}\,, \quad v_\mu = T^\sigma{}_{\sigma\mu}\,,$$
$$t_{\sigma\mu\nu} = \frac{1}{2}\left(T_{\sigma\mu\nu} + T_{\mu\sigma\nu}\right) + \frac{1}{6}\left(g_{\nu\sigma}v_\mu + g_{\nu\mu}v_\sigma\right) - \frac{1}{3}g_{\sigma\mu}v_\nu\,,$$

where $\epsilon_{\mu\nu\sigma\rho}$ is the totally anti-symmetric Levi-Civita symbol. From these we build the scalars

$$T_{\text{ax}} = a_\mu a^\mu\,, \quad T_{\text{vec}} = v_\mu v^\mu\,, \quad T_{\text{ten}} = t_{\sigma\mu\nu}t^{\sigma\mu\nu}\,,$$

and the torsion scalar is a linear combination

$$T = \frac{3}{2}T_{\text{ax}} + \frac{2}{3}T_{\text{ten}} - \frac{2}{3}T_{\text{vec}}\,.$$

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- If $c_1=-\frac{2}{3}$, $c_2=\frac{3}{2}$, $c_3=\frac{2}{3}$, the above action is equivalent to the TEGR one.

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- This theory contains parity-preserving quadratic form of the torsion with three free parameters.
- Perturbations around Minkowski shows that the unique stable Minkowski background that includes gravity is the TEGR case[2].

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$f(T)$ gravity

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Not equivalence between $f(\mathring{R})$ and $f(T)$

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- **Strongly coupling problem?** By performing Minkowski perturbations, one only finds new modes at 4th order in the perturbation (J. Beltrán Jiménez, A. Golovnev, T. Koivisto and H. Veermäe, [arXiv:2004.07536])

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- Other theories related to the boundary term such as $-T + f(B)$ gravity.

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- Compatible with Solar System tests.
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The resulting field equations must be at most second order in terms of derivatives of the tetrad fields (or equivalently second order in terms of metric tensor derivatives).

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Condition 3

The field equations must be covariant under local Lorentz transformations.

Conditions for the theory

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Any number of contractions of the irreducible parts of the torsion tensor will result in second order field equations. This means that an infinite number of terms can be formed in Teleparallel gravity that give rise to second order field equations. However, it is unclear how physical such higher order contributions will be.

Teleparallel Horndeski gravity

- Taking quadratic contractions of the torsion tensor, the most general Lagrangian of Teleparallel gravity satisfying the conditions turns out to be $f(T_{\text{ax}}, T_{\text{vec}}, T_{\text{ten}})$[5]

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Possible independent scalars

$$I_2 = v^{\mu}\phi_{;\mu}\,, \quad J_1 = a^{\mu}a^{\nu}\phi_{;\mu}\phi_{;\nu}\,, \quad J_3 = v_{\sigma}t^{\sigma\mu\nu}\phi_{;\mu}\phi_{;\nu}\,,$$
$$J_5 = t^{\sigma\mu\nu}t_{\sigma}{}^{\bar{\mu}}{}_{\nu}\phi_{;\mu}\phi_{;\bar{\mu}}\,, \quad J_6 = t^{\sigma\mu\nu}t_{\sigma}{}^{\bar{\mu}\bar{\nu}}\phi_{;\mu}\phi_{;\nu}\phi_{;\bar{\mu}}\phi_{;\bar{\nu}}\,,$$
$$J_8 = t^{\sigma\mu\nu}t_{\sigma\mu}{}^{\bar{\nu}}\phi_{;\nu}\phi_{;\bar{\nu}}\,, \quad J_{10} = \epsilon^{\mu}{}_{\nu\rho\sigma}a^{\nu}t^{\alpha\rho\sigma}\phi_{;\mu}\phi_{;\alpha}\,.$$

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Teleparallel Horndeski action

$$\mathcal{S}_{\rm Tele-deski} = \int d^4x \Big[\mathcal{L}_{\text{Horndeski}} + \mathcal{L}_{\text{Tele}}\Big] e = \int d^4x \Big[\sum_{i=2}^{5} \mathcal{L}_i + G_{\text{Tele}}\Big] e\,,$$

where[6]

$$\begin{aligned}
G_{\text{Tele}} &= G_{\text{Tele}}(\phi, X, T, T_{\text{ax}}, T_{\text{vec}}, I_2, J_1, J_3, J_5, J_6, J_8, J_{10})\,,\\
\mathcal{L}_2 &= G_2(\phi, X)\,, \quad \mathcal{L}_3 = G_3(\phi, X)\mathring{\Box}\phi\,,\\
\mathcal{L}_4 &= G_4(\phi, X)\left(-T + B\right) + G_{4,X}(\phi, X)\left[\left(\mathring{\Box}\phi\right)^2 - \phi_{;\mu\nu}\phi^{;\mu\nu}\right]\,,\\
\mathcal{L}_5 &= G_5(\phi, X)\mathcal{G}_{\mu\nu}\phi^{;\mu\nu} - \frac{1}{6}G_{5,X}(\phi, X)\Big[\left(\mathring{\Box}\phi\right)^3 + 2\phi_{;\mu}{}^{\nu}\phi_{;\nu}{}^{\alpha}\phi_{;\alpha}{}^{\mu}\\
&\quad - 3\phi_{;\mu\nu}\phi^{\mu\nu}\left(\mathring{\Box}\phi\right)\Big]\,.
\end{aligned}$$

[6] S. Bahamonde, K. F. Dialektopoulos and J. Levi Said, Phys. Rev. D **100** (2019) no.6, 064018.

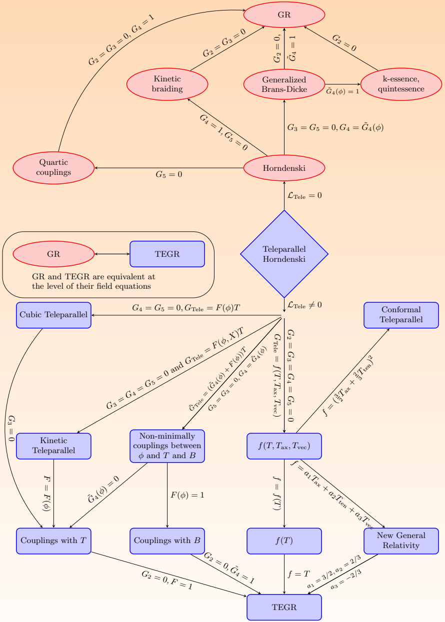

FIG. 1: Relationship between Teleparallel Horndenski and various theories.

Outline

How to work with different geometric symmetries

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FLRW tetrad compatible with cosmological symmetries in the Weitzenböck gauge

$$e^a{}_\mu = \text{diag}(N(t), a(t), a(t), a(t))$$
$$\rightarrow ds^2 = N(t)^2 - a(t)^2(dx^2 + dy^2 + dz^2)\,.$$

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Antisymmetric field equations

Important point: the tetrad showed in the last slide in the Weitzenböck gauge solves all the antisymmetric field equations for any Teleparallel gravitational theory

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Spherical coordinates

Be careful here: In spherical coordinates (t, r, θ, ϕ), the tetrad in the Weitzenböck gauge looks more complicated (off-diagonal terms appear):

$$e^a{}_\mu = \begin{pmatrix} N(t) & 0 & 0 & 0 \\ 0 & a(t)\sin(\theta)\cos(\phi) & ra(t)\cos(\theta)\cos(\phi) & -ra(t)\sin(\theta)\sin(\phi) \\ 0 & a(t)\sin(\theta)\sin(\phi) & ra(t)\cos(\theta)\sin(\phi) & ra(t)\sin(\theta)\cos(\phi) \\ 0 & a(t)\cos(\theta) & -ra(t)\sin(\theta) & 0 \end{pmatrix}.$$

Cosmological perturbations in TG

- In a metrical theory, one perturbs the FLRW metric in the scalar-vector-tensor decomposition form:

$$\delta g_{\mu\nu} = \begin{bmatrix} -2\varphi & a\left(\partial_i\mathcal{B}+\mathcal{B}_i\right) \\ a\left(\partial_i\mathcal{B}+\mathcal{B}_i\right) & 2a^2\left(-\psi\delta_{ij}+\partial_i\partial_j h+2\partial_{(i}h_{j)}+\frac{1}{2}h_{ij}\right) \end{bmatrix}.$$

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- In TG, the zeroth-order is ${e^a}_\mu = \mathrm{diag}(N(t), a(t), a(t), a(t))$ with ${\omega^a}_{b\mu} = 0$. We perturb this tetrad reproducing the above metric:

$$\delta {e^A}_\mu = \begin{bmatrix} \varphi & a\left(\partial_i \beta + \beta_i\right) \\ {\delta^I}_i\left(\partial^i b + b^i\right) & a\delta^{Ii}\left(-\psi\delta_{ij} + \partial_i\partial_j h + 2\partial_{(i}h_{j)} + \frac{1}{2}h_{ij} + \epsilon_{ijk}\left(\partial^k\sigma + \sigma^k\right)\right) \end{bmatrix}.$$

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- The **metric has 10 d.o.f.** (4 scalars(1 each), 2 vectors(2 each), 1 tensor(2 each)) and the **tetrads 16 d.o.f.** (6 scalars(1 each), 4 vectors(2 each), 1 tensor(2 each)).

Background cosmology in $f(T,B)$ gravity

- The modified FLRW equations in $f(T,B)$ gravity are[7]

$$\begin{aligned} -3H^2(3f_B+2f_T)+3H\dot{f}_B-3\dot{H}f_B+\frac{1}{2}f &= \kappa^2\rho_m\,, \\ -\left(3H^2+\dot{H}\right)(3f_B+2f_T)-2H\dot{f}_T+\ddot{f}_B+\frac{1}{2}f &= -\kappa^2 p_m\,. \end{aligned}$$

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meaning that $c_T^2 = 1$ with a Planck mass run rate $\alpha_M = \frac{1}{H}\frac{\dot{f}_T}{f_T}$. Thus, $f_T < 0$ is required for stability issues.

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$$\Sigma = \frac{1}{2}\frac{G_{\rm eff}}{G}\left(1+\frac{\psi}{\varphi}\right)\,.$$

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Cosmological perturbations in $f(T,B)$ gravity

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- There are different branches having different $G_{\rm eff}$ depending on the form of f.
- For example for $f_{BB}+2f_{TB}+f_{TT}=0$ one finds $G_{\rm eff}=-G\frac{4}{3(f_T+12H^2f_{TB})}$. One can use these results to constrain models.

Cosmological perturbations in $f(T,B)$ gravity - H_0 tension

- $f(T)$ gravity model does not show tension on the H_0 that prevails in the ΛCDM cosmology, however, σ_8 tension persists(R. C. Nunes, JCAP **05** (2018), 052)

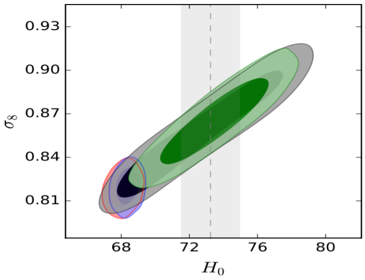

Figure 4. Parametric space in the plane H_0 - σ_8, where the regions in red (blue) show the constraints for ΛCDM model from CMB + BAO (CMB + BAO + H_0), respectively. The regions in black (green) show the constraints for $f(T)$ gravity from CMB + BAO (CMB + BAO + H_0), respectively. The vertical gray band corresponds to $H_0 = 73.24 \pm 1.74$ km s^{-1} Mpc^{-1}.

Background cosmology in Teleparallel scalar-tensor

- Teleparallel dark energy[9] (coupling like $\xi\phi^2 T$) exhibits a quintessence-like, dark-energy-dominated solution, or to the stiff dark-energy late-time attractor, similarly to standard quintessence. There is an additional late-time solution, in which dark energy behaves like a cosmological constant.

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- TG non-local cosmology with a term like $Tf(\mathring{\Box}^{-1}T)$ in the action is consistent with the present cosmological data obtained by SNe Ia + BAO + CC + H0 observations[11]

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Teleparallel Horndeski gravity - perturbations

- By considering tensorial perturbations only and after some cumbersome calculations, one gets the following wave equation

$$\ddot{h}_{ij} + (3 + \alpha_{\mathsf{M}})\, H\dot{h}_{ij} - (1 + \alpha_{\mathsf{T}})\, \frac{k^2}{a^2} h_{ij} = 0,$$

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where $\alpha_T = c_T^2 - 1$ and the speed of GW being equal to[12]

Speed of GW in Teleparallel Horndeski

$$c_T^2 = \frac{G_4 - X(\ddot{\phi}G_{5,X} + G_{5,\phi}) - G_{\rm Tele,T}}{G_4 - 2XG_{4,X} - X(H\dot{\phi}G_{5,X} - G_{5,\phi}) + 2XG_{\rm Tele,J_8} + \frac{1}{2}XG_{\rm Tele,J_5} - G_{\rm Tele,T}}\,.$$

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Reviving Horndeski using Teleparallel gravity

- For $G_{\rm Tele}=0$ (standard case), one gets that to achieve a theory consistent with the GW observations $c_T=1$, one requires $G_5(\phi,X)=\text{constant}$ and $G_4(\phi,X)=G_4(\phi)$. Hence, Horndeski gravity is highly constraint.

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Teleparallel Lagrangian respecting $c_T=1$ ($\alpha_T=0$)

$$\mathcal{L}=\tilde{G}_{\rm tele}(\phi,X,T,T_{\rm vec},I_2)+\sum_{i=2}^{4}\mathcal{L}_i+G_5(\phi)\mathcal{G}_{\mu\nu}\phi^{;\mu\nu}\,.$$

Outline

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- It is possible to formulate a theory which is equivalent to GR, and then, one can modify these equations to explain dark energy or inflation.
- One needs to be more careful than in Riemannian theories since the tetrad and spin connection form a pair that always need to be considered in a proper way to fulfill the symmetry condition to then solve the antisymmetric field equations.

Conclusions

- Two important TG theories: $f(T,B)$ (contains $f(\mathring{R})$) and Teleparallel Horndeski (contains many scalar-tensor theories).

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Conclusions

- Two important TG theories: $f(T,B)$ (contains $f(\mathring{R})$) and Teleparallel Horndeski (contains many scalar-tensor theories).
- TG cosmology can explain dark energy, alleviate H_0 tension and there are may interesting models with interesting features.
- There are many things totally unexplored in TG, so please go ahead!

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The main topics include:

- Extensions of General Relativity (metric affine gravity, Poincare gauge gravity, scalar/vector/tensor gravity, teleparallel gravity, massive gravity, bi-metric gravity, ...)
- Astrophysics in Extended Gravity (black holes, ordinary/neutron/boson/grava stars, gravitational waves, strings, wormholes, binary systems, ...)
- Cosmology in Extended Gravity (dynamical system analysis, observations and constraints, dark energy, dark matter, inflation, early universe, galaxies, ...)
- Beyond Lorentzian Geometry in Classical and Quantum Gravity (doubly/deformed relativity, standard model extension, Hamilton geometry, Finsler geometry, ...).

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