Teleparallel quintessence with a nonminimal coupling to a boundary term

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Outline

1. Introduction
   - General Relativity
   - Teleparallel gravity
2. Nonminimally coupled scalar fields
   - Nonminimally coupled scalar fields with the scalar curvature
   - Nonminimally coupled scalar fields to the scalar torsion
   - Teleparallel quintessence with a nonminimal coupling to B
3. Cosmology
   - Flat FRW
   - Nonminimal coupling purely to the boundary term
4. Conclusions

Outline

General Relativity

- Energy/matter $\neq 0 \iff$ Curved space-time.
- Metric tensor $\rightarrow$ measure angles and longitudes in curved space-times by $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$.
- Energy/matter $\rightarrow$ Energy-momentum tensor $\mathcal{T}^{\mu\nu}$.
- Curvature of the space-time $\rightarrow$ Riemaninan curvature tensor $R^\mu{}_{\nu\lambda\sigma}$. (function of $g_{\mu\nu}$)
- The most general connection which defines the parallel transportation is

Spin connection for GR

$$\text{Spin connection} = \Gamma^\lambda{}_{\mu\nu}\ (\text{Riemaniann tensor}) + \cancel{K_\mu{}^\lambda{}_\nu}\ (\overbrace{\text{torsion tensor}}^{=0 \text{ in GR}})$$

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- The Einstein-Hilbert action

$$S = \int \left[\frac{R}{2\kappa^2} + L_{\mathrm{m}} \right] \sqrt{-g}\, d^4x \,. \qquad (1)$$

where $\kappa = 8\pi G/c^4$ is a constant, R is the scalar curvature $R = g^{\mu\nu} R_{\mu\nu} = R^{\nu}{}_{\nu}$, which is a contraction of the Ricci tensor $R^{\lambda}{}_{\mu\lambda\nu}$. In addition, L_{m} describes any matter fields where $\mathcal{T}_{\mu\nu} := \frac{-2}{\sqrt{-g}} \frac{\delta(\sqrt{-g} L_m)}{\delta g^{\mu\nu}}$.

- This action arises to the Einstein's field equations which can be understood as

Einstein's field equation

$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa \mathcal{T}_{\mu\nu}$ (geometry = κ energy/matter)

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Why do we need another theory of gravity?

- G.R. has classical problems in cosmology ! (cosmological constant problem, coincidence problem, the phantom barrier, etc.)
- The Universe started with a Big Bang (point with infinity density and Temperature) and it is currently expanding in a accelerating rate → dark energy. (why?) GR does not explain this very well !

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Teleparallel equivalent of general relativity

- Weitzenböck noticed that it is always possible to define a connection $W_{\mu}{}^{\lambda}{}_{\nu}$ on a space such that is globally flat ($R^{\lambda}{}_{\mu\nu\sigma} \equiv 0$), therefore

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Teleparallel equivalent of general relativity

- The teleparallel action is formulated based on a gravitational scalar called the torsion scalar T

$$S = \int \left[-\frac{T}{2\kappa^2} + L_{\rm m} \right] e\, d^4x\,. \tag{3}$$

where $e = \det(e^{\mu}_{a}) = \sqrt{-g}$.

- The relationship between the scalar curvature R and the scalar torsion is

Relationship between R and T

$$R = -T + B\,. \tag{4}$$

The term B is a boundary term $\rightarrow$ Einstein-Hilbert action and the teleparallel action arise to the same field equations.

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Outline

Nonminimally coupled scalar fields with the scalar curvature

The first approach to nonminimally coupling a scalar field to the gravitational sector is to consider a coupling to the scalar curvature as follows

$$S = \int \left[\frac{R}{2\kappa^2} + \frac{1}{2}(\partial_\mu \phi \partial^\mu \phi + \xi R \phi^2) - V(\phi) + L_{\rm m} \right] \sqrt{-g}\, d^4x \,. \tag{5}$$

Here, ϕ is a scalar field, $V(\phi)$ the energy potential energy and ξ is a coupling constant.

Nonminimally coupled quintessence ($\xi = 0$)

- Nonminimally coupled quintessence corresponds to taking $\xi = 0$ in that Lagrangian $\rightarrow$ Late time accelerated expansion of the Universe and inflation
- However, simple models of scalar field inflation are becoming disfavoured by the latest Planck data.
- In addition, the effective equation of state must always satisfy $w_{\rm eff} > -1$ and require a very flat fine tuned potential in order to explain current cosmological observations.

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Standard quintessence models ($\xi = 0$) in Cosmology

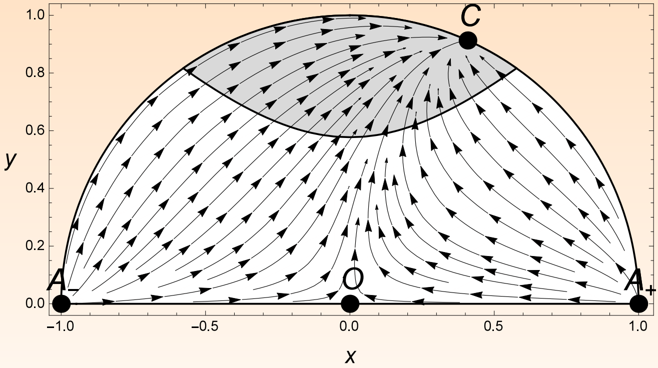

Figure: Phase space showing trajectories of standard quintessence models, for the particular parameter choice $w = 0$, $\lambda = 1$ (with $\chi = 0$). The point C is the late time accelerating attractor, with the shaded region indicating the region of acceleration.

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Nonminimally coupled scalar fields to the scalar torsion

An alternative approach has been to consider a scalar field nonminimally coupled to torsion. The following action is considered

$$S = \int \left[-\frac{T}{2\kappa^2} + \frac{1}{2}(\partial_\mu \phi \partial^\mu \phi - \xi T \phi^2) - V(\phi) + L_{\rm m} \right] e\, d^4x \,. \quad (6)$$

- Setting $\xi = 0$, the two theories again become equivalent due to the teleparallel equivalence.
- Phantom and quintessence type dynamics possible and Dynamical crossing of the phantom barrier.
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Teleparallel quintessence with a nonminimal coupling to a boundary term

With the aim of unifying both of the previous considered approaches, we proposed a more general action given by

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- This action was motivated by the relationship $R = -T + B$.
- Setting $\chi = -\xi \rightarrow$ nonminimal coupling to the scalar curvature R.
- Setting $\chi = 0 \rightarrow$ nonminimal coupling to the scalar torsion action T.
- $\xi = 0$ has not been studied yet!

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- By varying the action with respect to the tetrad and the scalar field we find the following field equations

$$-\left(\frac{2}{\kappa^2}+2\xi\phi^2\right)\left[e^{-1}\partial_\mu(eS_a{}^{\mu\nu})-E_a^\lambda T^\rho{}_{\mu\lambda}S_\rho{}^{\nu\mu}-\frac{1}{4}E_a^\nu T\right]$$
$$-E_a^\nu\left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-V(\phi)\right]+E_a^\mu\partial^\nu\phi\partial_\mu\phi-4(\xi+\chi)E_a^\rho S_\rho{}^{\mu\nu}\phi\partial_\mu\phi$$
$$-\chi\left[E_a^\nu\Box(\phi^2)-E_a^\mu\nabla^\nu\nabla_\mu(\phi^2)\right] = T_a^\nu. \quad (8)$$

$$\Box\phi+V'(\phi)=(\xi T+\chi B)\phi. \quad (9)$$

- Very large and complicated in general !
- However, due to the isotropy and homogeneity principles, the field equations in cosmology become easier.

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Cosmology

- We will consider the standard spatially flat Friedmann-Robertson-Walker (FRW) tetrad given by

$$e^a_\mu = \mathrm{diag}(1, a(t), a(t), a(t)), \tag{10}$$

- corresponding to a spatially flat FRW metric

$$ds^2 = dt^2 - a(t)^2(dx^2 + dy^2 + dz^2), \tag{11}$$

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- The energy momentum tensor of the matter sector is standard barotropic matter given by an isotropic perfect fluid

$$\mathcal{T}^{\mu}_{\nu} = \mathrm{diag}(\rho, -p, -p, -p). \tag{12}$$

Here, $\rho = \rho(t)$ and $p = p(t)$ are the matter energy density and pressure respectively.

- In cosmology, we use a fluid with barotropic equation of state $p = w\rho$, where w is a constant matter equation of state parameter.
- $w = 0 \rightarrow$ dust, $w = 1/3 \rightarrow$ radiation and $w = -1 \rightarrow$ cosmological constant (dark energy).

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Cosmology

- Inserting this FRW tetrad into the field equations give us

$$3H^2 = \kappa^2\left(\rho + \rho_\phi\right), \quad (13)$$

$$3H^2 + 2\dot{H} = -\kappa^2\left(p + p_\phi\right). \quad (14)$$

Here $H = \frac{\dot{a}}{a}$ is the Hubble parameter.

- We have defined the energy density and pressure of the scalar field as follows

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) - 3\xi H^2\phi^2 + 6\chi H\phi\dot{\phi}, \quad (15)$$

$$p_\phi = \frac{1}{2}(1-4\chi)\dot{\phi}^2 - V(\phi) + 2H\phi\dot{\phi}(2\xi + 3\chi) + 3H^2\phi^2(\xi + 8\chi^2) + 2\phi^2\dot{H}(\xi + 6\chi^2) + 2\chi\phi V'(\phi). \quad (16)$$

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Cosmology

In addition, the Klein-Gordon equation $\Box\phi + V'(\phi) = (\xi T + \chi B)\phi$ reduces to

$$\ddot{\phi} + 3H\dot{\phi} + 6(\xi H^2 + \chi(3H^2 + \dot{H}))\phi + V'(\phi) = 0. \quad (17)$$

In order to close our system, we need to specify the energy potential. Hereafter, we will consider $V(\phi) = V_0\, e^{-\lambda\kappa\phi}$.

Outline

Nonminimal coupling purely to the boundary term B

- Dynamical systems where $\xi = 0$ (purely coupling of the scalar field to the boundary term).
- Let us introduce the dimensionless variables

$$\sigma^2 = \frac{\kappa^2 \rho}{3H^2}, \quad x^2 = \frac{\kappa^2 \dot{\phi}^2}{6H^2}, \quad y^2 = \frac{\kappa^2 V}{3H^2}, \quad z = 2\sqrt{6}\kappa\chi\phi. \tag{18}$$

- $y > 0$ since $V(\phi) > 0$ and x or z can be positive or negative since $\dot{\phi}$ and χ can have any sign.
- The first Friedman equation written in these variables is simply

$$1 = \sigma^2 + x^2 + y^2 + xz. \tag{19}$$

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Dynamical system - Phase space

- The boundary of our phase space will be 3-Dimensional given by the constraint ($\rho \geq 0$)

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Dynamical system - Equations

- By replacing the dimensionless variables in the field equations we find

$$x' = -\frac{2\sqrt{6}y^2\lambda(xz-2) + (2x+z)\left(6x^2(4\chi+w-1) + 6x(w-1)z + 6y^2(w+1) - 6w + z^2 + 6\right)}{2\left(z^2+4\right)}, \tag{21}$$

$$y' = -\frac{y\left(\sqrt{6}\lambda\left(x\left(z^2+4\right) + 2y^2 z\right) + 4\left(3x^2(w-1) + 3x(w-1)z + 3y^2(w+1) - 3w - z^2 - 3\right)\right)}{2\left(z^2+4\right)} \tag{22}$$

$$z' = 12\chi x. \tag{23}$$

- Here, we defined $\lambda = -\frac{V'(\phi)}{\kappa V(\phi)}$, $N = \ln a$ and denote derivatives with respect to N by a prime.

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Dynamical system - Finite critical points

Point	(x, y, z)	Existence
O	$(0, 0, 0)$	$\forall\lambda,\ \chi$
$A_{\pm}$	$(\pm 1, 0, 0)$	$\chi = 0$
B	$\left(\sqrt{\frac{3}{2}}\frac{(1+w)}{\lambda}, \sqrt{\frac{3}{2}}\frac{\sqrt{(1+w)(1-w)}}{\lambda}, 0\right)$	$\chi = 0$ and $\lambda^2 > 3(1+w)$
D	$\left(0, \frac{\sqrt{\frac{3}{2}}\sqrt{\sqrt{w+1}\sqrt{4\lambda^2+9w+9}-3w-3}}{\lambda}, \frac{\sqrt{\frac{3}{2}}\left(\sqrt{w+1}\sqrt{4\lambda^2+9w+9}-3w-3\right)}{\lambda}\right)$	$\forall\lambda,\ \chi$
E	$(0, 1, 0)$	$\lambda = 0$ and $\chi \neq 0$

Table: Critical points of the autonomous system, along with the conditions for existence of the point. Points $A_{\pm}$ and B are quasi-stationary.

Dynamical system - Finite critical points

Point	w_{eff}	Acceleration	Eigenvalues	Stability
O	0	No	$\frac{3}{2}, -\frac{3}{4}\left(\sqrt{1-16\chi}+1\right), \frac{3}{4}\left(\sqrt{1-16\chi}-1\right)$	Saddle node
A_-	1	No	$3,\ 3+\sqrt{\frac{3}{2}}\lambda$	Unstable node: $\lambda > -\sqrt{6}$ Saddle node: otherwise
A_+	1	No	$3,\ 3-\sqrt{\frac{3}{2}}\lambda$	Unstable node: $\lambda < \sqrt{6}$ Saddle node: otherwise
B	0	No	$\frac{3}{4}+\frac{3\sqrt{24-7\lambda^2}}{4\lambda}, -\frac{3}{4}+\frac{3\sqrt{24-7\lambda^2}}{4\lambda}$	Stable node: $3<\lambda^2<24/7$ Stable spiral: $\lambda^2>24/7$
D	-1	Yes	$\Delta_1, \Delta_2, \Delta_3$	Stable spiral: $\chi>0$ Saddle point: $\chi<0$
E	-1	Yes	$-3,\ -\frac{3}{2}\left(\sqrt{1-8\chi}+1\right)$ $\frac{3}{2}\left(\sqrt{1-8\chi}-1\right)$	Stable spiral: $\chi>1/8$ Stable node: $0<\chi<1/8$ Saddle point: $\chi<0$

Table: Stability and eigenvalues and $w_{eff} = \frac{p+p_\phi}{\rho+\rho_\phi}$, assuming a (dark) matter equation of state $w=0$.

Dynamical system - Critical points at infinity

- The phase space is not compact so we must check whether there are critical points at infinity.
- We begin by introducing the compactified coordinates x_r, y_r and z_r like so

$$x_r = \frac{x}{\sqrt{1+r^2}}, \quad y_r = \frac{y}{\sqrt{1+r^2}}, \quad z_r = \frac{z}{\sqrt{1+r^2}}, \qquad (24)$$

where $r^2 = x^2 + y^2 + z^2$.
- We also define the quantity $\rho = \frac{r}{\sqrt{1+r^2}}$, so that $x_r^2 + y_r^2 + z_r^2 = \rho^2$. This means the dynamics at infinity will now be captured by taking the limit $\rho \to 1$.

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Dynamical system - Critical points at infinity

- We then make a further coordinate transformation, transforming the Poincaré variables into spherical polar coordinates as so

$$x_r = \rho\cos\theta\sin\varphi, \quad z_r = \rho\sin\theta\sin\varphi, \quad y_r = \rho\cos\varphi, \quad (25)$$

where the variables lie in the range $\rho \in [0, 1]$, $\theta \in [0, 2\pi]$ and since we are restricting ourselves to $y \geq 0$ the angle φ lies in the restricted range $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Dynamical system - Critical points at infinity

- Transforming our dynamical system into these new variables, in the limit $\rho \to 1$ we find the following

$$\rho' = 0, \tag{26}$$

$$\sqrt{1-\rho^2}\theta' = \sqrt{6}\lambda \cos\theta \cos\varphi \cot\varphi, \tag{27}$$

$$\sqrt{1-\rho^2}\varphi' = \sqrt{\frac{3}{2}}\lambda \cos\varphi \left(2\sin\theta\cos^2\varphi + \cos\theta\sin^2\varphi\right), \tag{28}$$

- Setting the right hand side of these equations equal to zero, we find that we must have $\cos\varphi = 0$, and hence the critical points are those at infinity which obey

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Dynamical system - Critical points at infinity

- Now we can use this to find an equation for θ'

$$\theta' = -24\chi \cos\theta \cot\theta(\sin\theta + \cos\theta) - 2\sin 2\theta + \frac{19}{4}\cos 2\theta + 6\cot\theta + \frac{17}{4}. \quad (30)$$

- Setting the right hand side of this equal to zero gives the critical points at infinity.
- At the critical points, the dark energy density parameter Ω_ϕ is divergent $\to$ unphysical critical points.
- Therefore we do not need to care about the critical points at the infinity !

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Cosmological implications

Nonminmal coupling purely with B (our model with $\xi = 0$)

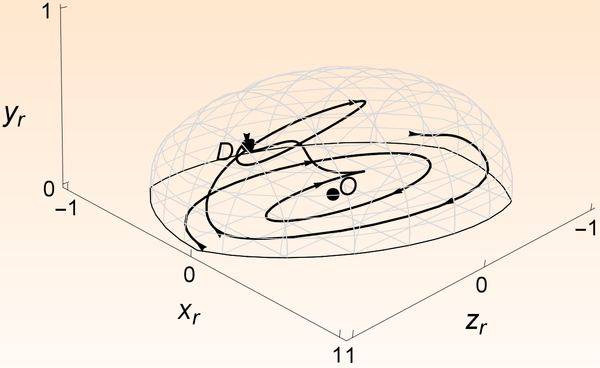

Figure: Phase space showing trajectories in Poincaré variables when $\chi = 1$, $\lambda = 2$ and $w = 0$. Point D is the global attractor.

Cosmological implications

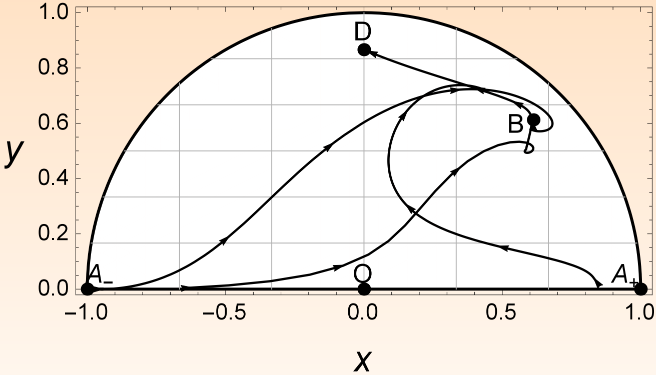

Figure: Phase space showing trajectories projected onto the $x - y$ plane when $\chi = 10^{-3}, \lambda = 2$ and $w = 0$. The points $A_\pm$ and B are quasi-stationary. Point D is again the global attractor.

Cosmological implications

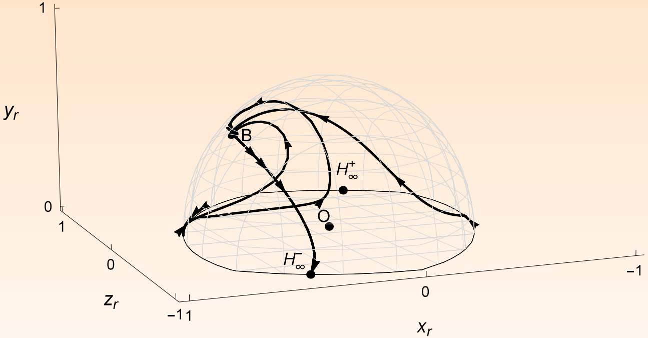

Figure: Phase space showing trajectories when $\chi = -10^{-3}, \lambda = 2$ and $w = 0$. Trajectories end at unphysical critical points lying at infinity.

Conclusions

- We proposed introducing a nonminimal coupling of a scalar field to both the torsion scalar T and the boundary term B.
- We analysed in detail the dynamics of the background cosmology when we have simply a coupling to the boundary term
- For a positive coupling, the system generically evolves to a late time dark energy dominated attractor, whose effective equation of state is exactly -1. This is independent of the potential, and thus requires absolutely no tuning of the potential to achieve this.
- While the system is evolving close to this late time attractor, the phantom barrier can be crossed, a scenario impossible without the presence of the coupling.

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References

S. Bahamonde and M. Wright, "Teleparallel quintessence with a nonminimal coupling to a boundary term," Phys. Rev. D **92** (2015) 8, 084034 [arXiv:1508.06580 [gr-qc]].

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