Modified teleparallel theories of gravity

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Work in collaboration with Christian Böhmer.

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Outline

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Teleparallel equivalent of general relativity

- A general connection which defines the parallel transportation is

Most general spin connection (Einstein-Cartan theory)

$$\overbrace{w_\mu{}^\lambda{}_\nu}^{\text{General spin connection}} = \overbrace{\Gamma^\lambda_{\mu\nu}}^{\text{related with curvature}} + \overbrace{\tilde{K}_\mu{}^\lambda{}_\nu}^{\text{related with torsion}} .$$

- G.R. assumes for simplicity that $\tilde{K}_\mu{}^\lambda{}_\nu \equiv 0$
- Weitzenböck noticed that it is always possible to define a connection $W_\mu{}^\lambda{}_\nu$ on a space such that is globally flat $(R^\lambda{}_{\mu\nu\sigma} \equiv 0) \Longrightarrow$ TEGR

UCL

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Geometrical differences

Torsion and curvature

Curvature $\Longrightarrow$ how the tangent spaces roll along the curve.
Torsion $\Longrightarrow$ how tangent spaces twist around a curve when they are parallel transported

How gravity is explained in both theories?

G.R. $\Longrightarrow$ Gravity $\Longrightarrow$ Curvature of space-time
TEGR $\Longrightarrow$ Gravity $\Longrightarrow$ Forces (Torsion) in a flat space-time.

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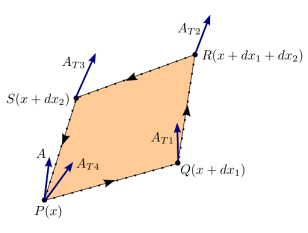

Figure: Transporting a vector in a closed trajectory creates a different vector

Torsion

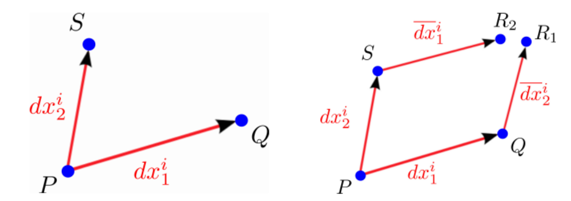

Figure: Transporting the vectors dx_1^i and dx_2^i towards the points S and Q creates a paralelogram which is not closed ($R_1 \neq R_2$)

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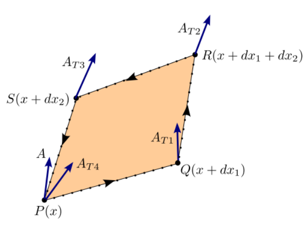

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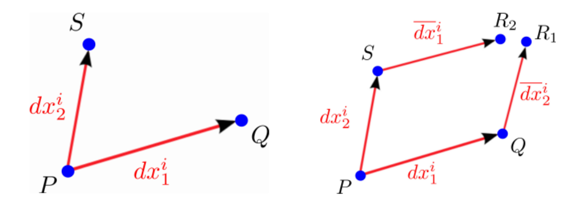

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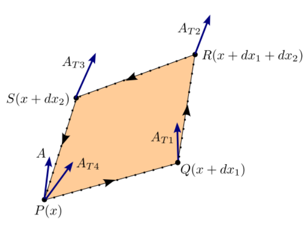

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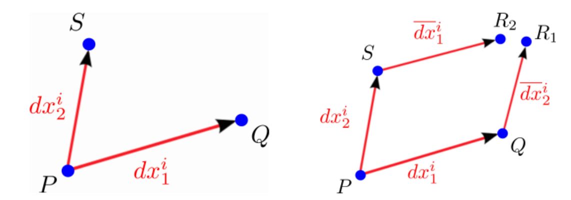

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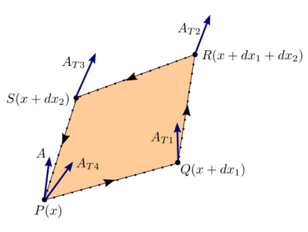

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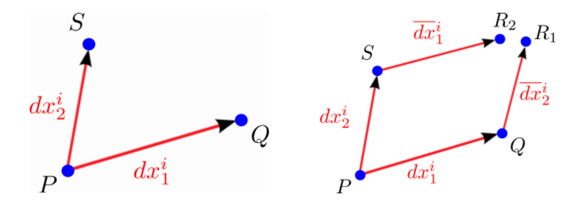

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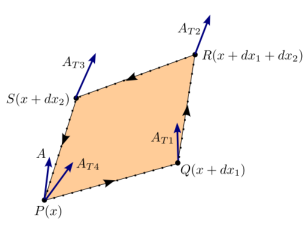

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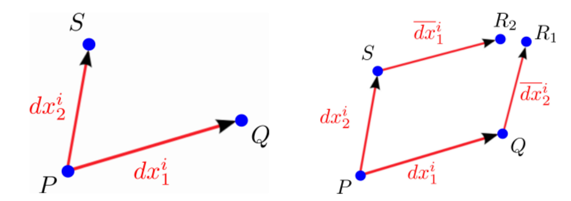

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Teleparallel equivalent of general relativity

- In TEGR, the dynamical variable is the tetrad field (or vierbein) .
- Tetrad fields are defined at each point of the manifold as a base of orthonormal vectors e^a_μ.

Metric and tetrad fields

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab} \,.$$

- Teleparallel gravity is an alternative formulation of gravity which is “equivalent” to general relativity (same field equations).

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- The teleparallel action is formulated based on a gravitational scalar called the torsion scalar T

TEGR action

$$S_{\mathrm{TEGR}} = \int \left[-\frac{T}{2\kappa} + L_{\mathrm{m}} \right] e \, d^4x \, .$$

Here, $e = \det(e^{\mu}_{a}) = \sqrt{-g}$ and $\kappa = 8\pi G$.

- The relationship between the scalar curvature R and the scalar torsion is

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- A well studied modification of GR is $f(R)$ gravity, which has the following action

$f(R)$ gravity action

$$S_{f(R)} = \int f(R)\sqrt{-g}\, d^4x \,.$$

- Here, f is an arbitrary (sufficiently smooth) function of the Ricci scalar.
- Ricci scalar depends on second derivatives of the metric tensor $\rightarrow$ **Fourth order theory.**

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- In analogy with $f(R)$ gravity, one can consider in the Teleparallel framework, the $f(T)$ gravity

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$$S_{f(T)} = \int f(T) e\, d^4x\,.$$

- The torsion scalar T depends on the first derivatives of the tetrads $\rightarrow$ **Second order theory.**
- T is not invariant under local LT $\Longrightarrow$ $f(T)$ **is also not invariant under local LT.**

Not equivalency between $f(R)$ and $f(T)$

Field equations of $f(T) \neq$ Field equations of $f(R)$

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$f(T, B)$ gravity

- Connection between both theories: $R = -T + B$
- Inspired by the later discussion, we define the action (Bahamonde et. al Phys.Rev. D92,10, 104042 (2015))

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$$S_{f(T,B)} = \int \left[\frac{1}{\kappa} f(T, B) + L_{\rm m} \right] e\, d^4x \,,$$

where f is a function of both of its arguments and $L_{\rm m}$ is a matter Lagrangian.

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$f(T,B)=f(T)$ gravity

- In order to recover $f(T)$ gravity, we simply set

Recovering $f(T)$ gravity

$$f(T,B)=f(T)\,.$$

- Doing this, we find

$$4e\Big[f_{TT}(\partial_\mu T)\Big]S_\nu{}^{\mu\lambda}+4e^a_\nu\partial_\mu(eS_a{}^{\mu\lambda})f_T$$
$$-4ef_TT^\sigma{}_{\mu\nu}S_\sigma{}^{\lambda\mu}-ef\delta^\lambda_\nu \;=\; 16\pi e\Theta^\lambda_\nu\,,$$

which, as expected, are the standard $f(T)$ field equations.

- This is the unique form of the function f which will give second order field equations

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- The Gauss-Bonnet term is a quadratic combination of the Riemann tensor and its contractions given by

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$$G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda}\,.$$

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Relationship between Gauss-Bonnet G and T_G

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- Inspired by the later discussion, we define the action (Bahamonde et. al arXiv:1606.05557 [gr-qc] (2016))

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$$S_{f(T,B,T_G,B_G)} = \int \left[\frac{1}{2\kappa} f(T,B,T_G,B_G) + L_{\rm m}\right] e\, d^4x\,.$$

- This action is very general. For instance, we can recover $f = f(T,T_G)$ gravity or $f(-T+B,-T_G+B_G) = f(R,G)$ gravity.

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Example: FRWL Cosmology with $k = +1$

- Since T, B, T_G and B_G are not invariant under local Lorentz transformations $\Longrightarrow$ One needs to be careful with the choice of the tetrad.
- The simplest tetrad field which yields the FRWL metric $ds^2 = -dt^2 + a(t)^2\left[\frac{1}{1-kr^2}dr^2 + d\Omega^2\right]$ is a diagonal one given by

$$e^a_\mu = \text{diag}\left(1, a(t)/\sqrt{1-kr^2}, a(t)r, a(t)r\sin\theta\right).$$

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- Since T, B, T_G and B_G are not invariant under local Lorentz transformations $\Longrightarrow$ One needs to be careful with the choice of the tetrad.
- The simplest tetrad field which yields the FRWL metric $ds^2=-dt^2+a(t)^2\left[\frac{1}{1-kr^2}dr^2+d\Omega^2\right]$ is a diagonal one given by

$$e^a_\mu = \text{diag}\Big(1, a(t)/\sqrt{1-kr^2}, a(t)r, a(t)r\sin\theta\Big)\,.$$

- PROBLEM: This tetrad is highly restrictive since it constraints the field equations with $f_{TT}=0$.

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Example: FRWL Cosmology with $k=+1$

- One approach to avoid this issue is by rotating the diagonal tetrad $\bar{e}^a_\mu = \Lambda^a{}_b e^b_\mu$, where

$$\Lambda^a{}_b = \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{R}(\alpha,\beta,\gamma) \end{pmatrix}.$$

Here, R is the 3-dimensional rotation in the tangent space parametrised by three Euler angles $\alpha=\theta-\frac{\pi}{2}$, $\beta=\phi$ and $\gamma=\gamma(r)$.

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Example: FRWL Cosmology with $k=+1$

- Using the rotated tetrad, the torsion scalar and the boundary term are

$$T=-\frac{4}{a^2}\Big(\frac{\sqrt{1-kr^2}}{r^2}\Big[r\gamma'\cos\gamma+\sin\gamma\Big]+\frac{1}{r^2}\Big)+6\frac{\dot{a}^2}{a^2}+2\frac{k}{a^2}\,,$$

$$B=-\frac{4}{a^2}\Big(\frac{\sqrt{1-kr^2}}{r^2}\Big[r\gamma'\cos\gamma+\sin\gamma\Big]+\frac{1}{r^2}\Big)+6\frac{\ddot{a}}{a}+12\frac{\dot{a}^2}{a^2}+8\frac{k}{a^2}\,.$$

- For the closed case $k=+1$, we need to set $\gamma(r)=-\operatorname{arcsinh}\left(\sqrt{1+r^2}\right)$ to have T and B position independent (also T_G and B_G).

Example: FRWL Cosmology with $k = +1$

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$$B = -\frac{4}{a^2}\Big(\frac{\sqrt{1-kr^2}}{r^2}\Big[r\gamma' \cos\gamma + \sin\gamma\Big] + \frac{1}{r^2}\Big) + 6\frac{\ddot{a}}{a} + 12\frac{\dot{a}^2}{a^2} + 8\frac{k}{a^2}\,.$$

- For the closed case $k = +1$, we need to set $\gamma(r) = -\operatorname{arcsinh}\left(\sqrt{1+r^2}\right)$ to have T and B position independent (also T_G and B_G).

Example: FRWL Cosmology with $k = +1$

- Using the rotated tetrad in a closed universe with the correct γ, the field equation are given by

$$f + \frac{6\dot{a}\dot{f}_B}{a} - \frac{6f_B\left(a\ddot{a}+2\dot{a}^2\right)}{a^2} - \frac{12\dot{a}^2 f_T}{a^2} - \frac{48 f_{B_G}\ddot{a}}{a^3} + \frac{48\dot{a}\dot{f}_{B_G}}{a^3} - \frac{24\left(\dot{a}^2-1\right)\dot{a}\dot{f}_{T_G}}{a^3} + \frac{24 f_{T_G}\left(\dot{a}^2-1\right)\ddot{a}}{a^3} = 2\kappa\,\rho\,,$$

$$f - \frac{48 f_{B_G}\ddot{a}}{a^3} - \frac{4\dot{a}\dot{f}_T}{a} + \frac{8(1-\dot{a}^2)\ddot{f}_{T_G}}{a^2} - \frac{6f_B\left(a\ddot{a}+2\dot{a}^2\right)}{a^2} - \frac{f_T\left(4a\ddot{a}+8\dot{a}^2-4\right)}{a^2} - \frac{16\dot{a}\ddot{a}\dot{f}_{T_G}}{a^2} + \frac{24 f_{T_G}\left(\dot{a}^2-1\right)\ddot{a}}{a^3} + \frac{16\ddot{f}_{B_G}}{a^2} + 2\ddot{f}_B = -2\kappa\,p\,.$$

- If $f(T,B,T_G,B_G) = \mathfrak{f}(-T+B,-T_G+B_G) = \mathfrak{f}(R,G)$ we recover the usual Gauss-Bonnet equations $\mathfrak{f}(R,G)$.

Example: FRWL Cosmology with $k=+1$

- Using the rotated tetrad in a closed universe with the correct γ, the field equation are given by

$$
\begin{aligned}
f+\frac{6\dot{a}\dot{f}_B}{a}-\frac{6f_B\left(a\ddot{a}+2\dot{a}^2\right)}{a^2}-\frac{12\dot{a}^2f_T}{a^2}-\frac{48f_{B_G}\ddot{a}}{a^3}+\frac{48\dot{a}\dot{f}_{B_G}}{a^3}\\
-\frac{24\left(\dot{a}^2-1\right)\dot{a}\dot{f}_{T_G}}{a^3}+\frac{24f_{T_G}\left(\dot{a}^2-1\right)\ddot{a}}{a^3}=2\kappa\,\rho\,,\\
f-\frac{48f_{B_G}\ddot{a}}{a^3}-\frac{4\dot{a}\dot{f}_T}{a}+\frac{8(1-\dot{a}^2)\ddot{f}_{T_G}}{a^2}-\frac{6f_B\left(a\ddot{a}+2\dot{a}^2\right)}{a^2}\\
-\frac{f_T\left(4a\ddot{a}+8\dot{a}^2-4\right)}{a^2}-\frac{16\dot{a}\ddot{a}\dot{f}_{T_G}}{a^2}+\frac{24f_{T_G}\left(\dot{a}^2-1\right)\ddot{a}}{a^3}\\
+\frac{16\ddot{f}_{B_G}}{a^2}+2\ddot{f}_B=-2\kappa\,p\,.
\end{aligned}
$$

- If $f(T,B,T_G,B_G)=\mathfrak{f}(-T+B,-T_G+B_G)=\mathfrak{f}(R,G)$ we recover the usual Gauss-Bonnet equations $\mathfrak{f}(R,G)$.

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Trace extension

We will now consider the later framework and we include the trace of the energy-momentum tensor to the action discussed before. This will give us the extended action

Trace and Gauss-Bonnet action extension

$$S_{f(T,B,T_G,B_G,\mathcal{T})} = \int \left[\frac{1}{2\kappa} f(T, B, T_G, B_G, \mathcal{T}) + L_{\rm m}\right] e\, d^4x\,,$$

where additionally f is a function of the trace of the energy-momentum tensor $\mathcal{T} = E_a^{\beta}\mathcal{T}_{\beta}^{a}$

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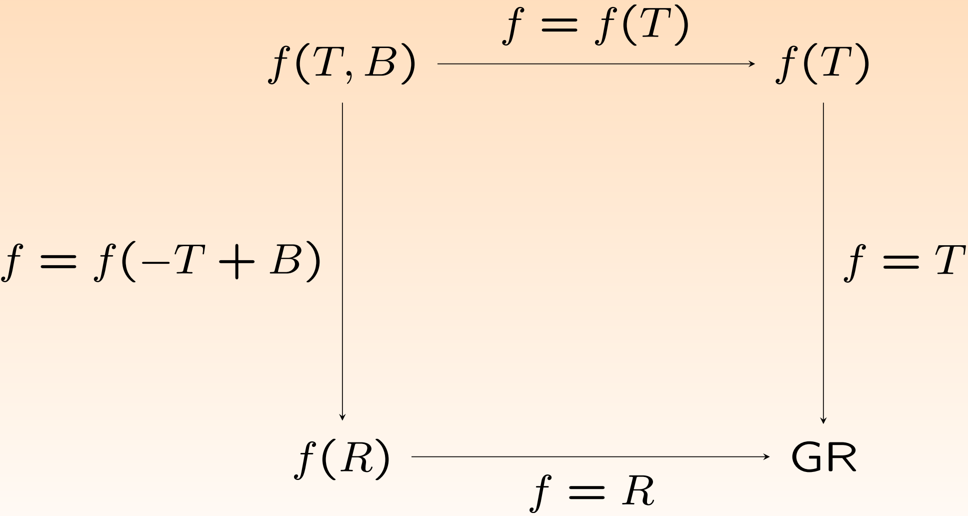
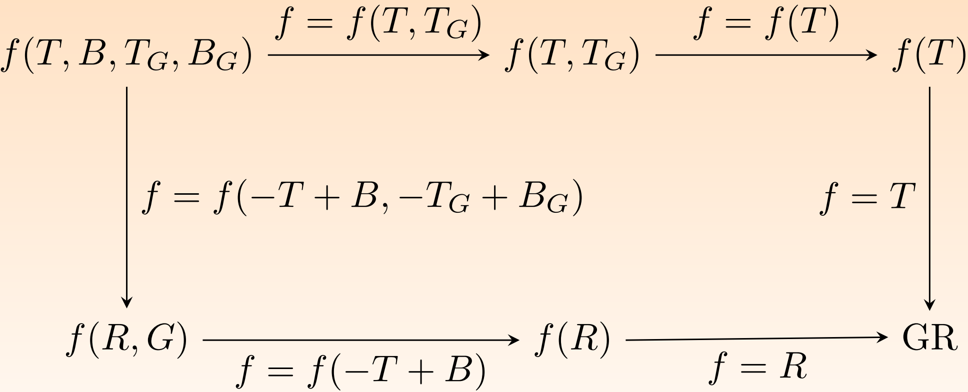
Outline

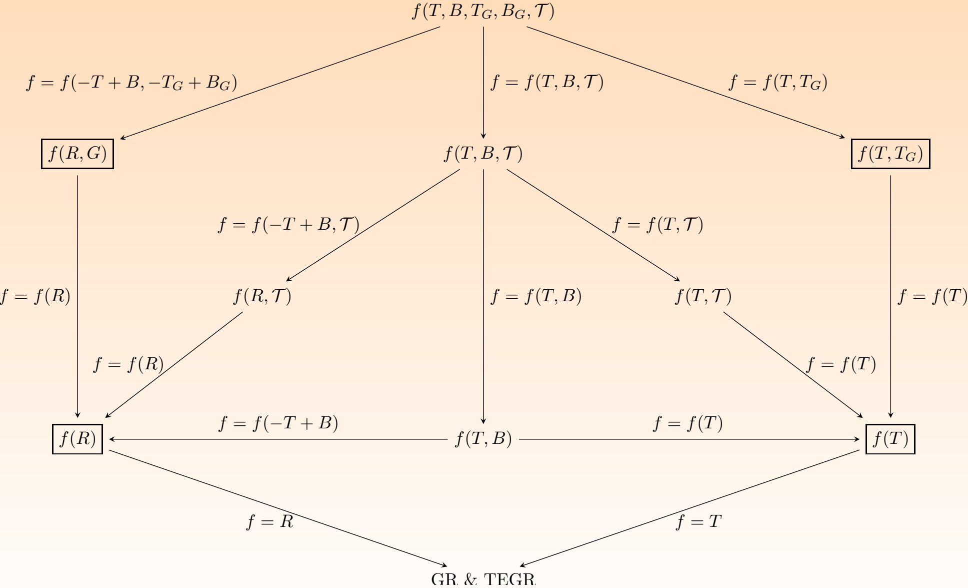
$f(T, B)$ diagram



$f(T, B, T_G, B_G)$ diagram



$f(T, B, T_G, B_G, \mathcal{T})$ diagram



Conclusions

- For many years now, an ever increasing number of modifications of GR has been considered. Many of these theories were considered in isolation in the past and their relationship with other similarly looking theories was only made implicitly.
- In our works, we explicitly found the corresponding teleparallel equivalent to well-known theories of gravity such as $f(R)$, $f(R, G)$, $f(R, \mathcal{T})$ and others.

Conclusions

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Bibliography

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