Metric-affine gravity and Black holes

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JCAP **01** (2022) no.01, 011; JCAP **04** (2022) no.04, 011; and 2210.05998
Jointly with Jorge Gigante Valcarcel.

東京工業大学
Tokyo Institute of Technology

Overview of the Talk

1. Introduction to Metric-affine gravity
   - Why modified gravity?
   - Basic geometrical quantities
2. Metric-Affine gravity
   - Curvature, torsion and nonmetricity
   - Dynamics
3. MAG models with dynamical torsion and nonmetricity
   - Weyl part of nonmetricity
   - Axial symmetry in Weyl-Cartan geometry
   - Spherical symmetry with Shears and Weyl (complete nonmetricity)

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- **Riemannian geometry:** The connection is the Levi-Civita one.

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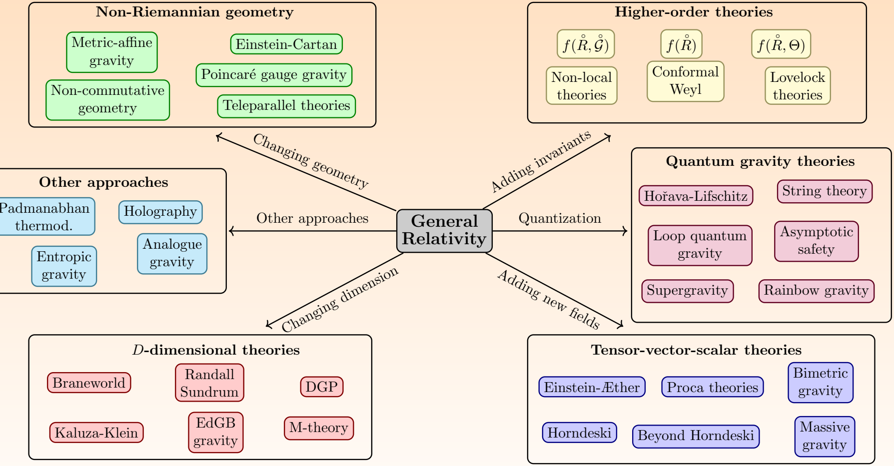
- Strong gravity regime needs to be tested;
- A good way to understand GR is to modify it;

How to modify it?



ure: Classification of theories of gravity. (S. Bahamonde et.al., "Teleparallel Gravity: From
ory to Cosmology," [arXiv:2106.13793 [gr-qc]].)

Fundamental variables and characteristic tensors

- In the most general metric-affine setting, the fundamental variables are a **metric** $g_{\mu\nu}$ (10 comp.) as well as the coefficients $\tilde{\Gamma}^{\rho}{}_{\mu\nu}$ (64 comp.) of an **affine connection**.

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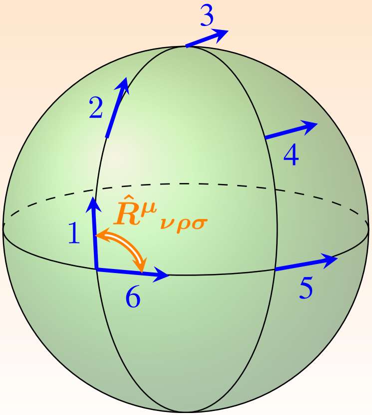
Curvature	$\tilde{R}^{\mu}{}_{\nu\rho\sigma} = \partial_\rho \tilde{\Gamma}^{\mu}{}_{\nu\sigma} - \partial_\sigma \tilde{\Gamma}^{\mu}{}_{\nu\rho} + \tilde{\Gamma}^{\mu}{}_{\tau\rho} \tilde{\Gamma}^{\tau}{}_{\nu\sigma} - \tilde{\Gamma}^{\mu}{}_{\tau\sigma} \tilde{\Gamma}^{\tau}{}_{\nu\rho}$
Torsion	$\tilde{T}^{\mu}{}_{\nu\rho} = \tilde{\Gamma}^{\mu}{}_{\rho\nu} - \tilde{\Gamma}^{\mu}{}_{\nu\rho}$
Nonmetricity	$\tilde{Q}_{\mu\nu\rho} = \tilde{\nabla}_\mu g_{\nu\rho} = \partial_\mu g_{\nu\rho} - \tilde{\Gamma}^{\sigma}{}_{\nu\mu} g_{\sigma\rho} - \tilde{\Gamma}^{\sigma}{}_{\rho\mu} g_{\nu\sigma}$

What does curvature geometrically represent?

Curvature tensor $\tilde{R}^{\alpha}{}_{\beta\mu\nu}$

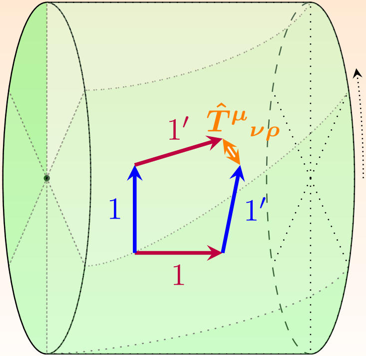
Rotation experienced by a vector when it is parallel transported along a closed curve



What does torsion geometrically represent?

Torsion tensor $\tilde{T}^{\alpha}{}_{\mu\nu}$

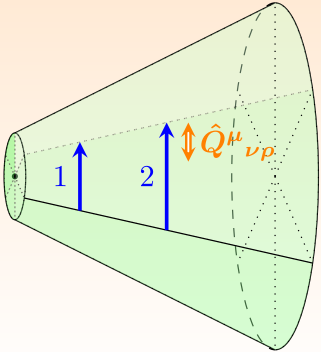
non-closure of the parallelogram formed when two infinitesimal vectors are parallel transported along each other.



What does non-metricity geometrically represent?

Non-metricity tensor $\tilde{Q}_{\alpha\mu\nu}$

measures how much the length and angle of vectors change as we parallel transport them, so in metric spaces the length of vectors is conserve



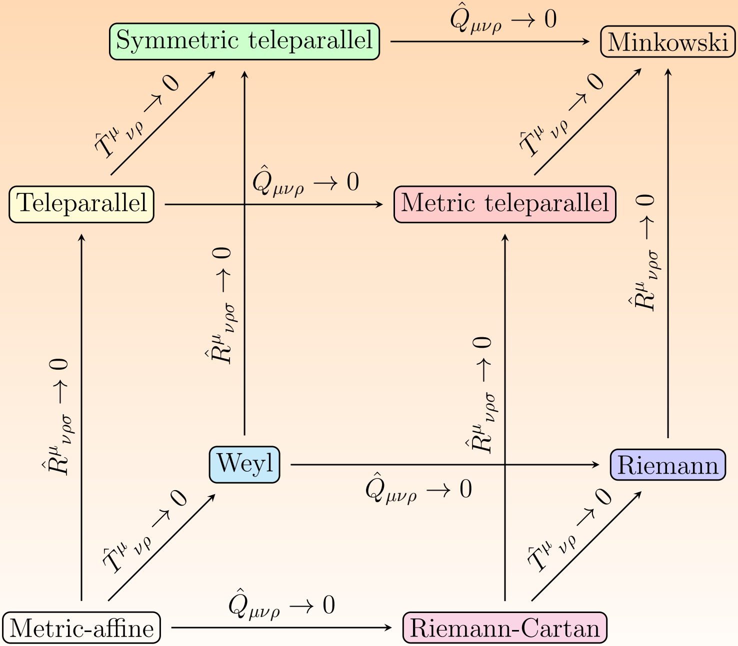


Figure: Classification of metric-affine geometries - Cube

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- The curvature tensor of an affinely connected metric space-time contains corrections provided by the presence of torsion and nonmetricity:(∇_ν just Levi-Civita)

$$\tilde{R}^{\lambda}{}_{\rho\mu\nu} = R^{\lambda}{}_{\rho\mu\nu} + \nabla_\mu N^{\lambda}{}_{\rho\nu} - \nabla_\nu N^{\lambda}{}_{\rho\mu} + N^{\lambda}{}_{\sigma\mu} N^{\sigma}{}_{\rho\nu} - N^{\lambda}{}_{\sigma\nu} N^{\sigma}{}_{\rho\mu}\,,$$

where

$$N^{\lambda}{}_{\mu\nu} = \frac{1}{2}\left(T^{\lambda}{}_{\mu\nu} - T_{\mu}{}^{\lambda}{}_{\nu} - T_{\nu}{}^{\lambda}{}_{\mu}\right) + \frac{1}{2}\left(Q^{\lambda}{}_{\mu\nu} - Q_{\mu}{}^{\lambda}{}_{\nu} - Q_{\nu}{}^{\lambda}{}_{\mu}\right)\,,$$

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- Furthermore, the latter also leads to the definition of three independent traces of this tensor, namely the Ricci and co-Ricci tensors:

$$\tilde{R}_{\mu\nu} = \tilde{R}^\lambda{}_{\mu\lambda\nu} \,, \quad \hat{R}_{\mu\nu} = \tilde{R}_\mu{}^\lambda{}_{\nu\lambda} \,,$$

  as well as the so-called homothetic curvature tensor $\tilde{R}^\lambda{}_{\lambda\mu\nu}$, which encodes the change of lengths of vectors provided by the trace part of nonmetricity under their parallel transport along closed loops.

- Due to torsion, this connection introduces modifications in the covariant derivative which indeed involves a change on its commutation relations when considering an arbitrary vector v^λ:

$$[\tilde{\nabla}_\mu, \tilde{\nabla}_\nu]\, v^\lambda = \tilde{R}^\lambda{}_{\rho\mu\nu}\, v^\rho + T^\rho{}_{\mu\nu}\, \tilde{\nabla}_\rho v^\lambda\,.$$

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- The change of lengths of a given vector k^μ as well as the change of angles between two unit timelike vectors $\hat{m}^\mu$ and $\hat{n}^\mu$, under a general parallel transport defined by a tangent vector V^μ, is proportional to the nonmetricity tensor:

$$V^\lambda \tilde{\nabla}_\lambda ||\mathbf{k}||^2 = V^\lambda Q_{\lambda\mu\nu} k^\mu k^\nu\,,$$

$$V^\lambda \tilde{\nabla}_\lambda \left(g_{\mu\nu}\hat{m}^\mu \hat{n}^\nu\right) = V^\lambda Q_{\lambda\mu\nu}\hat{m}^\mu \hat{n}^\nu - \frac{1}{2} V^\lambda Q_{\lambda\mu\nu}\left(\hat{m}^\mu \hat{m}^\nu + \hat{n}^\mu \hat{n}^\nu\right)\hat{m}^\rho \hat{n}_\rho\,.$$

Dynamics of metric-affine geometry

- Gravitational action with dynamical torsion and nonmetricity:

$$S = \int d^4x\sqrt{-g}\left[\mathcal{L}_m - \frac{1}{16\pi}\mathcal{L}_g(\tilde{\mathcal{R}}, \mathcal{T}, \mathcal{Q})\right] .$$

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- Correspondence between geometry and matter:

$$\begin{aligned}\frac{\delta S_g}{\delta e^a{}_\nu} &= 16\pi\theta_a{}^\nu\,,\\ \frac{\delta S_g}{\delta\omega^a{}_{b\nu}} &= 16\pi\Delta_a{}^{b\nu}\,.\end{aligned}$$

Here $\theta_a{}^\nu$ is the energy-momentum tensor (canonical) and $\Delta_a{}^{b\nu}$ is the hypermomentum density tensor.

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- $GL(4,R)$ group allows the definition of a large number of scalar invariants depending on the aforementioned tensors.

Dynamics of metric-affine geometry

- General quadratic gravitational action with dynamical torsion and nonmetricity:

$$
\begin{aligned}
S = \int d^4x\sqrt{-g}\Big\{\mathcal{L}_m + \frac{1}{16\pi}\Big[&-\tilde{R} + a_1\tilde{R}^2 + a_2\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\lambda\rho\mu\nu} + a_3\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\rho\lambda\mu\nu} \\
&+ a_4\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\mu\nu\lambda\rho} + a_5\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\lambda\mu\rho\nu} + a_6\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\mu\lambda\rho\nu} + a_7\tilde{R}_{\rho\lambda\mu\nu}\tilde{R}^{\mu\lambda\rho\nu} \\
&+ a_8\tilde{R}_{\mu\nu}\tilde{R}^{\mu\nu} + a_9\tilde{R}_{\mu\nu}\tilde{R}^{\nu\mu} + a_{10}\hat{R}_{\mu\nu}\hat{R}^{\mu\nu} + a_{11}\hat{R}_{\mu\nu}\hat{R}^{\nu\mu} + a_{12}\tilde{R}_{\mu\nu}\hat{R}^{\mu\nu} \\
&+ a_{13}\tilde{R}_{\mu\nu}\hat{R}^{\nu\mu} + a_{14}\tilde{R}^{\lambda}{}_{\lambda\mu\nu}\tilde{R}^{\rho}{}_{\rho}{}^{\mu\nu} + a_{15}\tilde{R}^{\lambda}{}_{\lambda\mu\nu}\tilde{R}^{\mu\nu} + a_{16}\tilde{R}^{\lambda}{}_{\lambda\mu\nu}\hat{R}^{\mu\nu} \\
&+ b_1 T_{\lambda\mu\nu}T^{\lambda\mu\nu} + b_2 T_{\lambda\mu\nu}T^{\mu\lambda\nu} + b_3 T^{\lambda}{}_{\lambda\nu}T^{\mu}{}_{\mu}{}^{\nu} + c_1 T_{\lambda\mu\nu}Q^{\mu\lambda\nu} \\
&+ c_2 T^{\lambda}{}_{\lambda\nu}Q^{\nu\mu}{}_{\mu} + c_3 T^{\lambda}{}_{\lambda\nu}Q^{\mu\nu}{}_{\mu} + d_1 Q_{\lambda\mu\nu}Q^{\lambda\mu\nu} + d_2 Q_{\lambda\mu\nu}Q^{\mu\lambda\nu} \\
&+ d_3 Q^{\lambda}{}_{\lambda\nu}Q^{\mu}{}_{\mu}{}^{\nu} + d_4 Q_{\nu}{}^{\lambda}{}_{\lambda}Q^{\nu\mu}{}_{\mu} + d_5 Q^{\lambda}{}_{\lambda\nu}Q^{\nu\mu}{}_{\mu}\Big]\Big\}\,.
\end{aligned}
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- Nonmetricity can be decomposed in the Weyl part plus a ”traceless” part:

$$Q_{\lambda\mu\nu} = g_{\mu\nu} W_\lambda + \not{Q}_{\lambda\mu\nu} \,.$$

where $W_\mu = \frac{1}{4} Q_{\mu\nu}{}^{\nu}$.

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- Quadratic gravitational action with dynamical torsion and nonmetricity in Weyl-Cartan geometry ($Q_{\lambda\mu\nu} = g_{\mu\nu} W_\lambda$ and $\not{Q}_{\lambda\mu\nu} = 0$)

$$S = \int d^4x \sqrt{-g} \Big\{ \mathcal{L}_{\rm m} + \frac{1}{64\pi} \Big[-4R - 6d_1 \tilde{R}_{\lambda[\rho\mu\nu]} \tilde{R}^{\lambda[\rho\mu\nu]} - 9d_1 \tilde{R}_{\lambda[\rho\mu\nu]} \tilde{R}^{\mu[\lambda\nu\rho]} + 8\, d_1 \tilde{R}_{[\mu\nu]} \tilde{R}^{[\mu\nu]} + \frac{1}{8} \left(32e_1 + 8e_2 + 17d_1\right) \tilde{R}^{\lambda}{}_{\lambda\mu\nu} \tilde{R}^{\rho}{}_{\rho}{}^{\mu\nu} - 7d_1 \tilde{R}_{[\mu\nu]} \tilde{R}^{\lambda}{}_{\lambda}{}^{\mu\nu} + 3\left(1 - 2a_2\right) T_{[\lambda\mu\nu]} T^{[\lambda\mu\nu]} \Big] \Big\} \,.$$

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- Absence of a general Birkhoff's theorem in MAG: new spherically and axially symmetric vacuum solutions with independent dynamical torsion and nonmetricity fields[1,2]

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Spherical symmetry

- Metric, torsion and nonmetricity in spherically symmetric space-times $(\#2 + \#8 + \#2 = \#12)$:

$$\mathcal{L}_\xi g_{\mu\nu} = \mathcal{L}_\xi T^\lambda{}_{\mu\nu} = \mathcal{L}_\xi W_\mu = 0 \implies \mathcal{L}_\xi \tilde{R}_{\lambda\rho\mu\nu} = 0$$

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- By solving these equations we find that torsion and nonmetricity behave as

$$\begin{aligned} T^t{}_{tr} &= a(r)\,, \quad T^r{}_{tr} = b(r)\,, \quad T^{\theta_k}{}_{t\theta_k} = f(r)\,, \quad T^{\theta_k}{}_{r\theta_k} = g(r) \\ T^{\theta_k}{}_{t\theta_l} &= e^{a\theta_k}\, e^b{}_{\theta_l}\, \epsilon_{ab}\, d(r)\,, \quad T^{\theta_k}{}_{r\theta_l} = e^{a\theta_k}\, e^b{}_{\theta_l}\, \epsilon_{ab}\, h(r)\,, \\ T^t{}_{\theta_k\theta_l} &= \epsilon_{kl}\, k(r)\, \sin\theta_1\,, \quad T^r{}_{\theta_k\theta_l} = \epsilon_{kl}\, l(r)\, \sin\theta_1\,, \\ W_\lambda &= (w_1(r), w_2(r), 0, 0)\,, \end{aligned}$$

whereas the metric is in the standard spherically symmetric form:

$$ds^2 = \Psi_1(r)\, dt^2 - \frac{dr^2}{\Psi_2(r)} - r^2 \left(d\theta_1^2 + \sin^2\theta_1 d\theta_2^2\right)\,.$$

Here, ϵ_{kl} is the Levi-Civita symbol in two dimensions.

Solution with dilations and spin

1. The solution for the metric behaves as Reissner-Nordström

$$g_{tt} = -1/g_{rr} \equiv \Psi(r) = 1 - \frac{2m}{r} + \frac{d_1\kappa_{\rm s}^2 - 4e_1\kappa_{\rm d,e}^2}{r^2}\,.$$

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3. Torsion sector:

$$\bar{\mathcal{S}}^a = -\frac{\kappa_{\rm s}}{r}(1,1,0,0)\,,$$

$$\bar{\mathcal{T}}_2^{abc} = \frac{\kappa_{\rm s}}{3r}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}\,.$$

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3. The solution is in vacuum and a charge κ_s appears (spin charge). Analogue to the case of Schwarzschild where the mass M appears.
4. We expect that the spin charge might be important in certain astrophysical scenarios such as: highly mangnetized neutron stars; supermassive black holes with endowed spin.

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  4. Do all particles in nature have different dilations? is this property important in particle physics?

Extension to axisymmetric space-times

- Metric, torsion and nonmetricity tensors in symmetric space-times:

$$\mathcal{L}_\xi g_{\mu\nu} = \mathcal{L}_\xi T^\lambda{}_{\mu\nu} = \mathcal{L}_\xi Q^\lambda{}_{\mu\nu} = 0 \implies \mathcal{L}_\xi \tilde{R}^\lambda{}_{\rho\mu\nu} = 0\,.$$

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- Stationary and axisymmetric space-times:

$$\#10 \to \#4 \begin{cases} ds^2 = \Psi_1(r,\vartheta)\,dt^2 - \frac{dr^2}{\Psi_2(r,\vartheta)} \\ -r^2\Psi_3(r,\vartheta)\Big[d\vartheta^2 + \sin^2\vartheta(d\varphi - \Psi_4(r,\vartheta)dt)^2\Big] \end{cases} ;$$

$$\#24 \Big\{ T^{\lambda}{}_{\mu\nu} = T^{\lambda}{}_{\mu\nu}(r,\vartheta)$$

$$\#4 \Big\{ W_\mu = (W_t(r,\vartheta), W_r(r,\vartheta), W_\vartheta(r,\vartheta), W_\varphi(r,\vartheta))\,.$$

Axisymmetric space-times - Kerr-Newmann de-Sitter

- Rotating Kerr-Newman metric structure[3]:

$$
\begin{aligned}
ds^2 = {} & \Psi(r,\vartheta)\, dt^2 - \frac{r^2 + a^2\cos^2\vartheta}{(r^2 + a^2\cos^2\vartheta)\,\Psi(r,\vartheta) + a^2\sin^2\vartheta}\, dr^2 \\
& - \left(r^2 + a^2\cos^2\vartheta\right) d\vartheta^2 + 2a\,(1 - \Psi(r,\vartheta))\sin^2\vartheta\, dt d\varphi \\
& - \sin^2\vartheta\left[r^2 + a^2 + a^2\,(1 - \Psi(r,\vartheta))\sin^2\vartheta\right] d\varphi^2\,,
\end{aligned}
$$

$$
\Psi(r,\vartheta) = 1 - \frac{\left[2mr + 4e_1(\kappa^2_{\mathrm{d,e}} + \kappa^2_{\mathrm{d,m}}) - d_1\kappa^2_{\mathrm{s}}\right]}{r^2 + a^2\cos^2\vartheta}\,.
$$

[3] S. Bahamonde and J. G. Valcarcel, JCAP **01** (2022) no.01, 011.

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$$

- Field strength tensors:

$$
\begin{aligned}
\bar{R}_{[\mu\nu]} &= \frac{1}{12}\varepsilon^{\lambda}{}_{\sigma\mu\nu}\nabla_\lambda\bar{S}^\sigma + \frac{1}{2}\nabla_\lambda\bar{t}^{\lambda}{}_{\mu\nu}\,; \quad \tilde{R}^{\lambda}{}_{\lambda\mu\nu} = 4\nabla_{[\nu}W_{\mu]}\,; \\
\bar{R}^{\lambda}{}_{[\mu\nu\rho]} &= \frac{1}{6}\varepsilon^{\lambda}{}_{\sigma[\rho\nu}\nabla_{\mu]}\bar{S}^\sigma + \nabla_{[\mu}\bar{t}^{\lambda}{}_{\rho\nu]} + \frac{1}{4}\varepsilon^{\lambda}{}_{\omega\sigma[\rho}\mathring{t}_1^{\sigma}{}_{\mu\nu]}\bar{S}^\omega \\
&\quad - \frac{1}{18}\varepsilon_{\sigma\mu\nu\rho}\mathring{T}_1^{\lambda}\bar{S}^\sigma\,.
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- Nonmetricity sector:(no approx.)

$$w_1(r,\vartheta) = \frac{\kappa_{\mathrm{d,e}} r - a\,\kappa_{\mathrm{d,m}} \cos\vartheta}{r^2 + a^2\cos^2\vartheta}\,, \quad w_3(r,\vartheta) = 0\,,$$

$$w_2(r,\vartheta) = -\frac{\kappa_{\mathrm{d,e}} r}{(r^2 + a^2\cos^2\vartheta)\Psi(r,\vartheta) + a^2\sin^2\vartheta}\,,$$

$$w_4(r,\vartheta) = \kappa_{\mathrm{d,m}}\left(\frac{r^2+a^2}{r^2 + a^2\cos^2\vartheta}\cos\vartheta - \gamma\right) - a\,\frac{\kappa_{\mathrm{d,e}} r \sin^2\vartheta}{r^2 + a^2\cos^2\vartheta}\,.$$

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- Torsion sector (decoupling limit between the spin and the orbital angular momentum $|a\kappa_{\mathrm{s}}| \ll 1$):

$$\bar{\mathcal{S}}^a = -\frac{\kappa_{\mathrm{s}}}{r}(1,1,0,0) + \mathcal{O}(a\kappa_{\mathrm{s}})\,,$$
$$\bar{\mathcal{T}}_2^{abc} = \frac{\kappa_{\mathrm{s}}}{3r}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 \end{pmatrix} + \mathcal{O}(a\kappa_{\mathrm{s}})\,.$$

Gravitational spin-orbit interaction

- We found a solution in the decoupling limit $a\kappa_s \ll 1$, which ensures that the Maxwell equation and closure conditions are fulfilled by the field strength tensors of torsion

$$\nabla_\lambda \tilde{R}^\lambda{}_{[\rho\mu\nu]} \quad = \quad \nabla_\mu \tilde{R}^{[\mu\nu]} = 0 \,, \quad \nabla_{[\sigma} \tilde{R}_{\lambda[\rho\mu\nu]]} = \nabla_{[\lambda} \tilde{R}_{[\mu\nu]]} = 0 \,.$$

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- **Gravitational spin-orbit interaction:**

$$\mathcal{H}_{\rm LS} = \frac{1}{m_{\rm e}^2 r}\frac{\partial V}{\partial r}\mathbf{L}\cdot\mathbf{S} \approx \frac{d_1}{2r}\frac{\partial g_{tt}}{\partial r} a\kappa_{\rm s}\cos\vartheta$$

- It is well known that the most general axisymmetric system in vacuum that can describe a BH type D in **GR** contains[4]:

Mass	M
Angular momentum	a
Taub-NUT charge	l
Acceleration	α

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- Further, one can add a cosmological constant Λ and a electric charge q_e and magnetic charge q_m.
- The solution in GR is called Plebanski-Damianski solution.

[4] J. F. Plebanski and M. Demianski, Annals Phys. **98** (1976), 98-127

- The Plebanski-Damianski metric was recently presented in an improved form with $\Lambda = 0$ in by Podolský and Vrátný (Phys. Rev. D **104** (2021), 084078), and it can be written as

$$ds^2 = \Omega^{-2}(r,\vartheta)\Big\{\Phi_1(r,\vartheta)\left[dt - \big(a\sin^2\vartheta + 2l(\chi - \cos\vartheta)\big)\,d\varphi\right]^2 - \frac{dr^2}{\Phi_1(r,\vartheta)} - \frac{d\vartheta^2}{\Phi_2(r,\vartheta)} - \Phi_2(r,\vartheta)\sin^2\vartheta\big[a\,dt - \big(r^2 + a^2 + l^2 + 2\chi a l\big)\,d\varphi\big]^2\Big\}\,.$$

where Φ_i, Ω are cumbersome functions depending on these parameters.

[5] S. Bahamonde, J. G. Valcarcel and L. Järv, JCAP **04** (2022) no.04, 011.

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$$- \frac{d\vartheta^2}{\Phi_2(r,\vartheta)} - \Phi_2(r,\vartheta)\sin^2\vartheta\big[a\,dt - \big(r^2 + a^2 + l^2 + 2\chi a l\big)\,d\varphi\big]^2\Big\}\,.$$

  where Φ_i, Ω are cumbersome functions depending on these parameters.
- We just found this new form with the cosmological constant[5] with $\Phi_1(r,\vartheta) = \frac{Q(r)}{\rho^2(r,\vartheta)}\,, \quad \Phi_2(r,\vartheta) = \frac{P(\vartheta)}{\rho^2(r,\vartheta)}$, and $\rho^2(r,\vartheta) = r^2 + (a\cos\vartheta + l)^2$. Here, $Q(r), \Omega(\vartheta)$ include the PD quantities.

[5] S. Bahamonde, J. G. Valcarcel and L. Järv, JCAP **04** (2022) no.04, 011.

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$$w_1(r,\vartheta) = \frac{\kappa_{\rm d,e} r - \kappa_{\rm d,m}(a\cos\vartheta + l)}{r^2 + (a\cos\vartheta + l)^2}\,, \quad w_2(r,\vartheta) = -\,\frac{\kappa_{\rm d,e} r - \kappa_{\rm d,m}\,(a\gamma + l)}{Q(r)}\,,$$

$$w_3(r,\vartheta) = -\,\kappa_{\rm d,m}\sqrt{K(\vartheta) - \left(\frac{\cot\vartheta - \gamma\csc\vartheta}{P(\vartheta)}\right)^2}\,,$$

$$w_4(r,\vartheta) = \kappa_{\rm d,m}\left[\frac{\left(r^2 + a^2 - l^2\right)\cos\vartheta + al\sin^2\vartheta + 2\chi l\,(a\cos\vartheta + l)}{r^2 + (a\cos\vartheta + l)^2} - \gamma\right] - \frac{\kappa_{\rm d,e} r\left[a\sin^2\vartheta + 2l\,(\chi - \cos\vartheta)\right]}{r^2 + (a\cos\vartheta + l)^2}\,,$$

$$\bar{T}^{\vartheta}{}_{\varphi t} = -\,\bar{T}^{\varphi}{}_{\vartheta t}\sin^2\vartheta = -\,\bar{T}^{\vartheta}{}_{\varphi r}\frac{Q(r)}{\rho^2(r,\vartheta)} = \bar{T}^{\varphi}{}_{\vartheta r}\frac{Q(r)}{\rho^2(r,\vartheta)}\sin^2\vartheta = \frac{\kappa_s\sin\vartheta}{r} + \mathcal{O}(x_i\kappa_s)\,.$$

- Similarly as electromagnetism, the torsion behaves as a Coulomb-like quantity depending on a spin charge κ_s and the non-metricity on the dilation charge κ_d.

Nonmetricity decomposition

- Nonmetricity can be decomposed in the Weyl part plus a ”traceless” part:

$$Q_{\lambda\mu\nu} = g_{\mu\nu} W_\lambda + \not{Q}_{\lambda\mu\nu} \,.$$

where

$$W_\mu = \frac{1}{4} Q_{\mu\nu}{}^{\nu} \,,$$

$$\not{Q}_{\lambda\mu\nu} = g_{\lambda(\mu}\Lambda_{\nu)} - \frac{1}{4} g_{\mu\nu}\Lambda_\lambda + \frac{1}{3}\varepsilon_{\lambda\rho\sigma(\mu}\Omega_{\nu)}{}^{\rho\sigma} + q_{\lambda\mu\nu} \,,$$

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- We defined a vector, and two traceless and pseudotraceless tensors

$$\Lambda_\mu = \frac{4}{9}\left(Q^{\nu}{}_{\mu\nu} - W_\mu\right) \,,$$

$$\Omega_\lambda{}^{\mu\nu} = -\left[\varepsilon^{\mu\nu\rho\sigma} Q_{\rho\sigma\lambda} + \varepsilon^{\mu\nu\rho}{}_{\lambda}\left(\frac{3}{4}\Lambda_\rho - W_\rho\right)\right] \,,$$

$$q_{\lambda\mu\nu} = Q_{(\lambda\mu\nu)} - g_{(\mu\nu}W_{\lambda)} - \frac{3}{4} g_{(\mu\nu}\Lambda_{\lambda)} \,,$$

The traceless part of nonmetricity and shears

- If this quantity is different to zero, when we parallel transport a vector, not only its norm changes but also its angle.

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- It is invariant under shears transformations.
- Shears: Deformations without changing the volume.
- Up to now, there are not exact solutions with shears in MAG.

MAG theory with shears

- Let us first consider a simple model where torsion is not propagating and the traceless part of nonmetricity is dynamical:

$$S=\frac{1}{16\pi}\int d^4x\sqrt{-g}\Big[-R+2f_1\tilde{R}_{(\lambda\rho)\mu\nu}\tilde{R}^{(\lambda\rho)\mu\nu}\\ +2f_2\left(\tilde{R}_{(\mu\nu)}-\hat{R}_{(\mu\nu)}\right)\left(\tilde{R}^{(\mu\nu)}-\hat{R}^{(\mu\nu)}\right)\Big]\,,$$

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- As can be seen, the propagation of the nonmetricity field described in the action is carried out by the symmetric part of the curvature tensor and its symmetric contraction:

$$\tilde{R}^{(\lambda\rho)}{}_{\mu\nu}=\tilde{\nabla}_{[\nu}Q_{\mu]}{}^{\lambda\rho}+\frac{1}{2}T^{\sigma}{}_{\mu\nu}Q_{\sigma}{}^{\lambda\rho}\,,$$
$$\tilde{R}_{(\mu\nu)}-\hat{R}_{(\mu\nu)}=\tilde{\nabla}_{(\mu}Q^{\lambda}{}_{\nu)\lambda}-\tilde{\nabla}_{\lambda}Q_{(\mu\nu)}{}^{\lambda}-Q^{\lambda\rho}{}_{\lambda}Q_{(\mu\nu)\rho}+Q_{\lambda\rho(\mu}Q_{\nu)}{}^{\lambda\rho}$$
$$+T_{\lambda\rho(\mu}Q^{\lambda\rho}{}_{\nu)}\,,$$

which in turn constitute deviations from the third Bianchi of GR.

Propagation of the traceless part

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which in turn constitute deviations from the third Bianchi of GR.

Spherical symmetry with nonmetricity and torsion

- Metric, torsion and nonmetricity in spherically symmetric space-times ($\#2 + \#8 + \#12 = \#22$):

$$\mathcal{L}_\xi g_{\mu\nu} = \mathcal{L}_\xi T^\lambda{}_{\mu\nu} = \mathcal{L}_\xi Q_{\alpha\mu\nu} = 0 \implies \mathcal{L}_\xi \tilde{R}_{\lambda\rho\mu\nu} = 0$$

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- Nonmetricity now contains all the 12 dof:

$$\begin{gathered}
Q_{ttt} = q_1(r)\,, \quad Q_{trr} = q_2(r)\,, \quad Q_{ttr} = q_3(r)\,, \\
Q_{t\vartheta\vartheta} = Q_{t\varphi\varphi}\csc^2\vartheta = q_4(r)\,, \quad Q_{rtt} = q_5(r)\,, \quad Q_{rrr} = q_6(r)\,, \\
Q_{rtr} = q_7(r)\,, \quad Q_{r\vartheta\vartheta} = Q_{r\varphi\varphi}\csc^2\vartheta = q_8(r)\,, \\
Q_{\vartheta t\vartheta} = Q_{\varphi t\varphi}\csc^2\vartheta = q_9(r)\,, \quad Q_{\vartheta r\vartheta} = Q_{\varphi r\varphi}\csc^2\vartheta = q_{10}(r)\,, \\
Q_{\vartheta t\varphi} = -Q_{\varphi t\vartheta} = q_{11}(r)\sin\vartheta\,, \quad Q_{\vartheta r\varphi} = -Q_{\varphi r\vartheta} = q_{12}(r)\sin\vartheta\,,
\end{gathered}$$

whereas the metric and torsion are the same as before.

How to find a solution with all of these dof?

- We are only interested in the traceless part of $Q_{\alpha\mu\nu}$ (containing shears), so that:

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q_1(r) &= \frac{\Psi_1(r)}{r^2}\left(r^2 q_2(r)\Psi_2(r) + 2q_4(r)\right) ,\\
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  2. We imposed $N_{[\lambda\rho]\mu} = 0$ which is equivalent to $T_{\lambda\mu\nu} = Q_{[\mu\nu]\lambda}$:
     $\rightarrow$ Shear transformations in the general linear group involves the part of the anholonomic connection describing a shear current or charge to take values in the **symmetric traceless part** of the Lie algebra.

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  3. We demand the corresponding torsion and nonmetricity scalars of the solution to be regular.

How to find a solution with all of these dof?

- We are only interested in the traceless part of $Q_{\alpha\mu\nu}$ (containing shears), so that:
  1. We eliminate the Weyl part of nonmetricity $W_\mu = \frac{1}{4} Q_{\mu\nu}{}^{\nu} = 0$ by setting

$$
\begin{aligned}
q_1(r) &= \frac{\Psi_1(r)}{r^2}\left(r^2 q_2(r)\Psi_2(r) + 2q_4(r)\right) \,, \\
q_5(r) &= \frac{\Psi_1(r)}{r^2}\left(r^2 q_6(r)\Psi_2(r) + 2q_8(r)\right) \,.
\end{aligned}
$$

  2. We imposed $N_{[\lambda\rho]\mu} = 0$ which is equivalent to $T_{\lambda\mu\nu} = Q_{[\mu\nu]\lambda}$:
     $\rightarrow$ Shear transformations in the general linear group involves the part of the anholonomic connection describing a shear current or charge to take values in the **symmetric traceless part** of the Lie algebra.
  3. We demand the corresponding torsion and nonmetricity scalars of the solution to be regular.
- After following these three steps we end up with 2 dof (metric)+ 5 dof (torsion/nonmetricity) which is only 7 dof.

New solution only with shears

- By plugging these conditions in the field equations, there are several branches but only one has solutions with dynamical shears. This branch involves the constants of the theory as

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- The form of q_i and t_i is involved. One component of nonmetricity is arbitrary! (problem!)
- The metric behaves as

$$ds^2 = \Psi_1(r)\, dt^2 - \frac{dr^2}{\Psi_2(r)} - r^2 \left(d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 \right) \, .$$

with

$$\Psi_1(r) = \Psi_2(r) = 1 - \frac{2m}{r} - \frac{2 f_1 \kappa_{\mathrm{sh}}^2}{r^2} \, ,$$

where κ_{sh} is interpreted as a new charge, ”shear charge”.

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- The action of the full model is

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S = \frac{1}{64\pi}\int\Big[&-4R - 6d_1\tilde{R}_{\lambda[\rho\mu\nu]}\tilde{R}^{\lambda[\rho\mu\nu]} - 9d_1\tilde{R}_{\lambda[\rho\mu\nu]}\tilde{R}^{\mu[\lambda\nu\rho]}\\
&+2d_1\left(\tilde{R}_{[\mu\nu]}+\hat{R}_{[\mu\nu]}\right)\left(\tilde{R}^{[\mu\nu]}+\hat{R}^{[\mu\nu]}\right)+18d_1\tilde{R}_{\lambda[\rho\mu\nu]}\tilde{R}^{(\lambda\rho)\mu\nu}\\
&-3d_1\tilde{R}_{(\lambda\rho)\mu\nu}\tilde{R}^{(\lambda\rho)\mu\nu}+6d_1\tilde{R}_{(\lambda\rho)\mu\nu}\tilde{R}^{(\lambda\mu)\rho\nu}+2\left(2e_1-f_1\right)\tilde{R}^{\lambda}{}_{\lambda\mu\nu}\tilde{R}^{\rho}{}_{\rho}{}^{\mu\nu}\\
&+8f_1\tilde{R}_{(\lambda\rho)\mu\nu}\tilde{R}^{(\lambda\rho)\mu\nu}-2f_1\left(\tilde{R}_{(\mu\nu)}-\hat{R}_{(\mu\nu)}\right)\left(\tilde{R}^{(\mu\nu)}-\hat{R}^{(\mu\nu)}\right)\\
&+3\left(1-2a_2\right)T_{[\lambda\mu\nu]}T^{[\lambda\mu\nu]}\Big]d^4x\sqrt{-g}\,.
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& - 3d_1 \tilde{R}_{(\lambda\rho)\mu\nu} \tilde{R}^{(\lambda\rho)\mu\nu} + 6d_1 \tilde{R}_{(\lambda\rho)\mu\nu} \tilde{R}^{(\lambda\mu)\rho\nu} + 2\left(2e_1 - f_1\right) \tilde{R}^{\lambda}{}_{\lambda\mu\nu} \tilde{R}^{\rho}{}_{\rho}{}^{\mu\nu} \\
& + 8f_1 \tilde{R}_{(\lambda\rho)\mu\nu} \tilde{R}^{(\lambda\rho)\mu\nu} - 2f_1 \left(\tilde{R}_{(\mu\nu)} - \hat{R}_{(\mu\nu)} \right) \left(\tilde{R}^{(\mu\nu)} - \hat{R}^{(\mu\nu)} \right) \\
& + 3\left(1 - 2a_2\right) T_{[\lambda\mu\nu]} T^{[\lambda\mu\nu]} \Big] d^4x \sqrt{-g} \,.
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- When traceless part of nonmetricity is zero, the above action coincides with the first one.

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- In this case, all nonmetricity components are fully set by the field equations (remember that in the shear case, one component was free)
- The solution gives us the following metric

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having the three possible charges: spin, dilation and shear.

- On the other hand, the solution can also be trivially generalised to include the cosmological constant and Coulomb electromagnetic fields with electric and magnetic charges q_e and q_m, which are decoupled from torsion under the assumption of the minimal coupling principle. This natural extension is then described by a Reissner-Nordström-de Sitter-like geometry

$$\Psi(r) = 1 - \frac{2m}{r} + \frac{d_1 \kappa_{\rm s}^2 - 4e_1 \kappa_{\rm d}^2 - 2f_1 \kappa_{\rm sh}^2 + q_e^2 + q_m^2}{r^2} + \frac{\Lambda}{3} r^2 \,,$$

which turns out to represent **the broadest family of static and spherically symmetric black hole solutions obtained in MAG** so far.

Conclusions and what do to next

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- The general solution contains the three fundamental charges (spin,dilation and shear) and the mass which constitute the most general spherically symmetric solution with all the possible intrinsic geometrical properties of matter.

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  3. What is the role of dilations/shears in particle physics?