Black holes solutions in metric-affine gravity with dynamical torsion and nonmetricity

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JCAP **01** (2022) no.01, 011; JCAP **04** (2022) no.04, 011.

東京工業大学
Tokyo Institute of Technology

Outline

Overview of the Talk

General Relativity - Assumptions

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- **Locality**

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- Strong gravity regime needs to be tested;
- A good way to understand GR is to modify it;

How to modify GR?

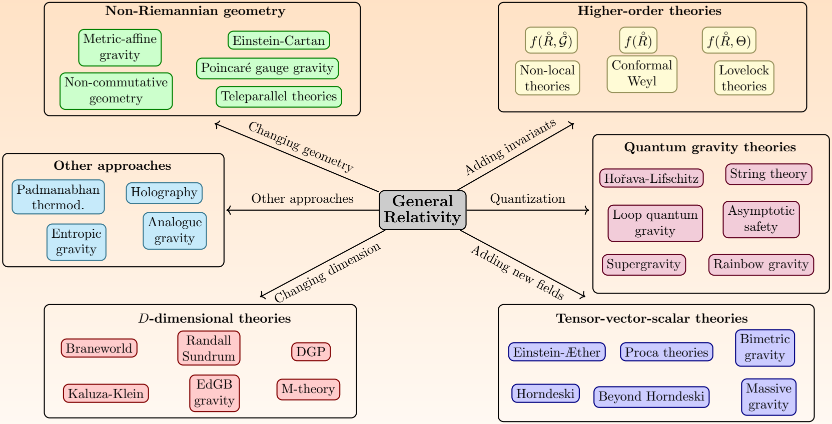

Figure: Classification of theories of gravity. (S. Bahamonde et.al., “Teleparallel Gravity: From Theory to Cosmology,” [arXiv:2106.13793 [gr-qc]].)

Fundamental variables and characteristic tensors

- In the most general metric-affine setting, the fundamental variables are a **metric** $g_{\mu\nu}$ (10 comp.) as well as the coefficients $\tilde{\Gamma}^{\rho}{}_{\mu\nu}$ (64 comp.) of an **affine connection.**

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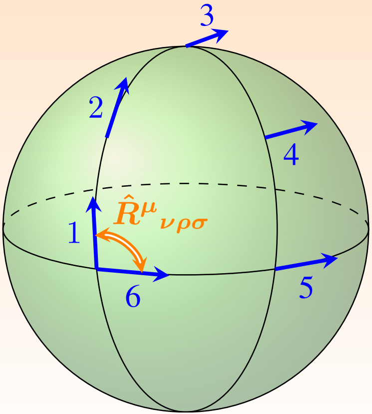
Curvature	$\tilde{R}^{\mu}{}_{\nu\rho\sigma} = \partial_{\rho}\tilde{\Gamma}^{\mu}{}_{\nu\sigma} - \partial_{\sigma}\tilde{\Gamma}^{\mu}{}_{\nu\rho} + \tilde{\Gamma}^{\mu}{}_{\tau\rho}\tilde{\Gamma}^{\tau}{}_{\nu\sigma} - \tilde{\Gamma}^{\mu}{}_{\tau\sigma}\tilde{\Gamma}^{\tau}{}_{\nu\rho}$
Torsion	$T^{\mu}{}_{\nu\rho} = \tilde{\Gamma}^{\mu}{}_{\rho\nu} - \tilde{\Gamma}^{\mu}{}_{\nu\rho}$
Nonmetricity	$Q_{\mu\nu\rho} = \tilde{\nabla}_{\mu}g_{\nu\rho} = \partial_{\mu}g_{\nu\rho} - \tilde{\Gamma}^{\sigma}{}_{\nu\mu}g_{\sigma\rho} - \tilde{\Gamma}^{\sigma}{}_{\rho\mu}g_{\nu\sigma}$

- Tildes: General connection; Curvature/connection without tildes are computed with the Levi-Civita.

What does curvature geometrically represent?

Curvature tensor $\tilde{R}^{\alpha}{}_{\beta\mu\nu}$

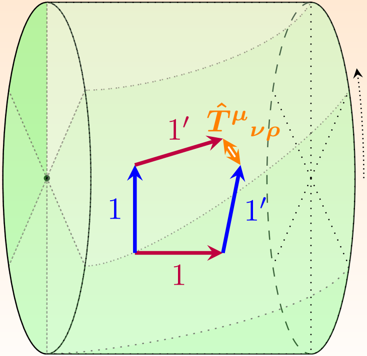
Rotation experienced by a vector when it is parallel transported along a closed curve



What does torsion geometrically represent?

Torsion tensor $\tilde{T}^{\alpha}{}_{\mu\nu}$

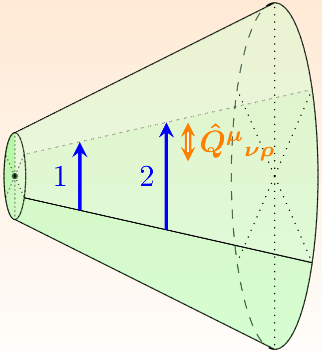
non-closure of the parallelogram formed when two infinitesimal vectors are parallel transported along each other.



What does non-metricity geometrically represent?

Non-metricity tensor $\tilde{Q}_{\alpha\mu\nu}$

measures how much the length and angle of vectors change as we parallel transport them, so in metric spaces the length of vectors is conserve



- **Riemann-Cartan geometry** ($\tilde{Q}_{\alpha\mu\nu} = 0$): If non-metricity vanishes, the metric satisfies the metric-compatibility condition $\tilde{\nabla}_{\mu} g_{\alpha\beta} = 0$. Poincaré grvity assumes this geometry.

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- **General Teleparallel geometry** ($\tilde{R}_{\alpha\mu\nu\beta} = 0$): In the case of vanishing curvature, the connection is flat.

- **Riemannian geometry** ($\tilde{T}^{\alpha}{}_{\mu\nu} = 0, \tilde{Q}_{\alpha\mu\nu} = 0$): The connection is symmetric and metric compatible, leading to $\tilde{\Gamma}^{\rho}{}_{\mu\nu} = \Gamma^{\rho}{}_{\mu\nu}$. GR and the majority of the theories are here.

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- **Symmetric Teleparallel geometry** ($\tilde{R}_{\alpha\mu\nu\beta} = 0, \tilde{T}^{\alpha}_{\mu\nu} = 0$): Both torsion tensor and curvature are zero and the gravitational interactions are only mediated through non-metricity.

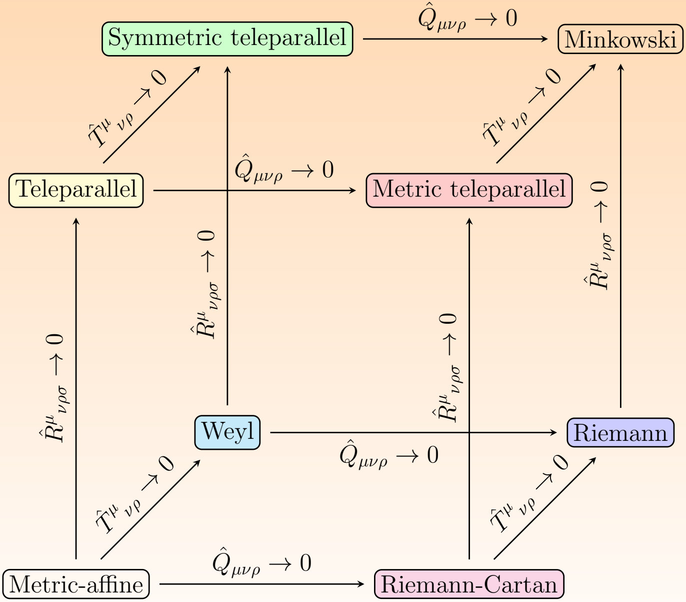

Figure: Classification of metric-affine geometries - Cube

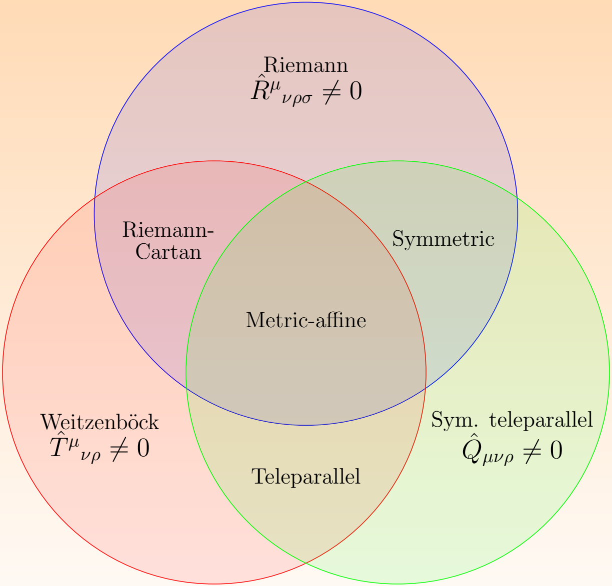

Figure: Classification of metric-affine geometries

Tetrads and the spin connection

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Metric and tetrads

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where $\eta_{ab} = \mathrm{diag}(1, -1, -1, -1)$ denotes the Minkowski metric.

Tetrads fields

The vectors $(\mathbf{E}_A)$ form an orthonormal basis of the tangent space, i.e.,

$$g(\mathbf{E}_a, \mathbf{E}_b) = g_{\mu\nu} {E_a}^{\mu} {E_b}^{\nu} = \eta_{ab} \,.$$

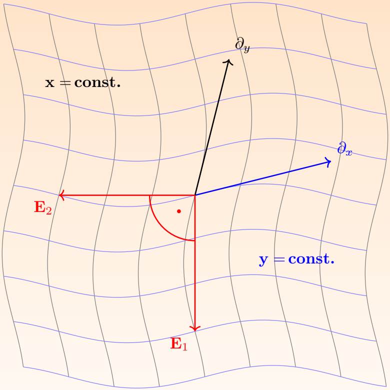

Figure: Graphical representation of the tetrad, reduced to a 2 dimensional model manifold. A coordinate basis (∂_x, ∂_y) of the tangent space is, by

Spin connection and tetrads

- The frame coefficients $E_a{}^\mu$ are also required in order to calculate the coefficients $\tilde{\Gamma}^\mu{}_{\nu\rho}$ of the affine connection from the spin connection $\tilde{\omega}^a{}_{b\mu}$ via

$$\tilde{\Gamma}^\rho{}_{\mu\nu} = E_a{}^\rho \left(\partial_\nu e^a{}_\mu + \tilde{\omega}^a{}_{b\nu} e^b{}_\mu \right) ,$$

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$$\tilde{\Gamma}^{\rho}{}_{\mu\nu} = E_a{}^{\rho}\left(\partial_{\nu} e^{a}{}_{\mu} + \tilde{\omega}^{a}{}_{b\nu} e^{b}{}_{\mu}\right) ,$$

- This is the unique affine connection satisfying the so-called “tetrad postulate”

$$\partial_{\mu} e^{a}{}_{\nu} + \tilde{\omega}^{a}{}_{b\mu} e^{b}{}_{\nu} - \tilde{\Gamma}^{\rho}{}_{\nu\mu} e^{a}{}_{\rho} = 0 .$$

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- One advantage of the formulation in terms of a tetrad and spin connection, is the fact that the curvature, torsion and non-metricity become properties of the spin connection only, and are independent of the choice of the tetrad.

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- One advantage of the formulation in terms of a tetrad and spin connection, is the fact that the curvature, torsion and non-metricity become properties of the spin connection only, and are independent of the choice of the tetrad.
- Then, we can define the curvature, torsion and nonmetricity as:

$$\tilde{R}^a{}_{b\mu\nu} := \partial_\mu \tilde{\omega}^a{}_{b\nu} - \partial_\nu \tilde{\omega}^a{}_{b\mu} + \tilde{\omega}^a{}_{c\mu}\tilde{\omega}^c{}_{b\nu} - \tilde{\omega}^a{}_{c\nu}\tilde{\omega}^c{}_{b\mu}\,,$$

$$\tilde{T}^a{}_{\mu\nu} := \partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu + \tilde{\omega}^a{}_{b\mu} e^b{}_\nu - \tilde{\omega}^a{}_{b\nu} e^b{}_\mu\,,$$

$$\tilde{Q}_{\mu ab} := -\eta_{ac}\tilde{\omega}^c{}_{b\mu} - \eta_{cb}\tilde{\omega}^c{}_{a\mu}\,.$$

Overview of the Talk

Trinity of gravity

As mentioned before, we can split the connection as

$$\tilde{\Gamma}^{\rho}{}_{\mu\nu} := \Gamma^{\rho}{}_{\mu\nu} + K^{\rho}{}_{\mu\nu} + L^{\rho}{}_{\mu\nu} := \Gamma^{\rho}{}_{\mu\nu} + D^{\rho}{}_{\mu\nu}\,, \tag{2}$$

where

$$\Gamma^{\mu}{}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma}\left(\partial_{\nu} g_{\sigma\rho} + \partial_{\rho} g_{\nu\sigma} - \partial_{\sigma} g_{\nu\rho}\right)\,, \quad \text{Levi Civita connection}$$

$$K^{\mu}{}_{\nu\rho} = \frac{1}{2}\left(T_{\nu}{}^{\mu}{}_{\rho} + T_{\rho}{}^{\mu}{}_{\nu} - T^{\mu}{}_{\nu\rho}\right)\,, \quad \text{Contortion tensor}$$

$$L^{\mu}{}_{\nu\rho} = \frac{1}{2}\left(Q^{\mu}{}_{\nu\rho} - Q_{\nu}{}^{\mu}{}_{\rho} - Q_{\rho}{}^{\mu}{}_{\nu}\right)\,, \quad \text{Disformation tensor}\,.$$

Trinity of gravity - curvature tensor

- The curvature becomes

$$\tilde{R}^{\mu}{}_{\nu\rho\sigma} = R^{\mu}{}_{\nu\rho\sigma} + \nabla_{\rho} D^{\mu}{}_{\nu\sigma} - \nabla_{\sigma} D^{\mu}{}_{\nu\rho} + D^{\mu}{}_{\tau\rho} D^{\tau}{}_{\nu\sigma} - D^{\mu}{}_{\tau\sigma} D^{\tau}{}_{\nu\rho} \,.$$

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- Now, by contracting the curvature tensor to obtain the Ricci scalar $\tilde{R} = g^{\mu\nu} \tilde{R}^{\rho}{}_{\mu\rho\nu}$ we find

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Ricci scalar decomposition

$$\tilde{R} = R + \Big(T + 2\nabla_{\mu}(\sqrt{-g} T^{\rho}{}_{\rho}{}^{\mu}) \Big) + \Big(Q + \nabla_{\mu} Q^{\mu\nu}{}_{\nu} - \nabla_{\nu} Q_{\mu}{}^{\mu\nu} \Big) + C$$

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with

$$T := T^{\rho\lambda\kappa} T_{\rho\lambda\kappa} + 2T^{\rho\lambda\kappa} T_{\kappa\rho\lambda} - 4T_{\rho}{}^{\kappa}{}_{\kappa} T^{\rho\lambda}{}_{\lambda} \,, \quad \text{Torsion scalar} \,,$$

$$Q := -\frac{1}{4} Q_{\alpha\beta\gamma} Q^{\alpha\beta\gamma} + \frac{1}{2} Q_{\alpha\beta\gamma} Q^{\beta\alpha\gamma} + \frac{1}{4} Q_{\alpha} Q^{\alpha} - \frac{1}{2} Q_{\alpha} \bar{Q}^{\alpha} \,, \quad \text{Nonmetricity scalar} \,,$$

$$C := 2(Q_{\kappa\rho\lambda} T^{\lambda\kappa\rho} + Q_{\rho}{}^{\sigma}{}_{\sigma} T^{\rho\kappa}{}_{\kappa} - Q^{\sigma}{}_{\sigma\rho} T^{\rho\kappa}{}_{\kappa}) \,.$$

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$$S_{\rm GR} = \int \left[-\frac{1}{2\kappa^2}R + L_{\rm m}\right]\sqrt{-g}\,d^4x\,.$$

where $\kappa^2 = 8\pi G$ and $L_{\rm m}$ is any matter Lagrangian.

- The Einstein's field equations are obtained by taking variations w/r to the metric: $\boxed{R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} = \kappa^2 T_{\mu\nu}}$.

Trinity of gravity - Teleparallel equivalent of GR

- Teleparallel equivalent of GR (TEGR) assumes zero curvature and zero nonmetricity so that

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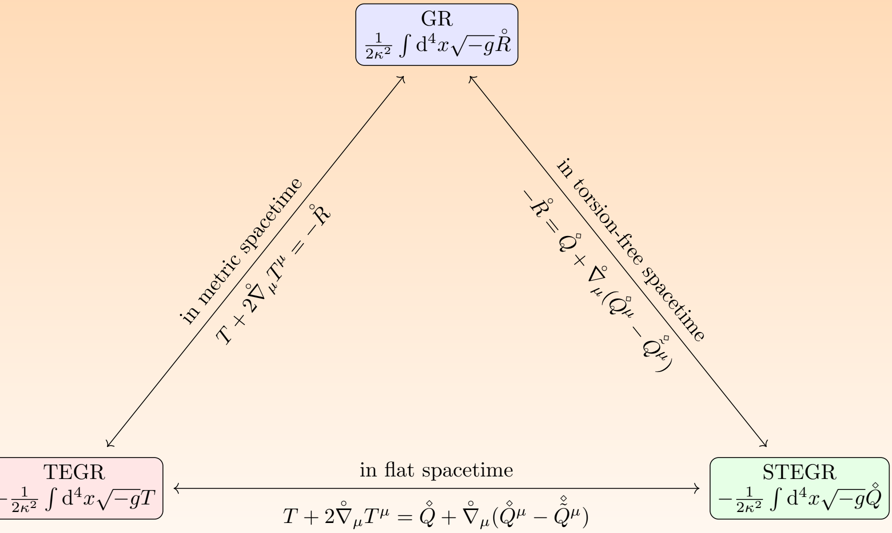

ure: Geometrical trinity of gravity (S. Bahamonde et.al., “Teleparallel Gravity: From Theory osmology,” [arXiv:2106.13793 [gr-qc]].)

Overview of the Talk

Gauge formalism of Poincaré gauge gravity

- Poincaré gauge gravity assumes zero nonmetricity $Q_{\alpha\mu\nu} = \tilde{\nabla}_\alpha g_{\mu\nu} = 0$ and a manifold with curvature and torsion.

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- Then, a gauge connection of the Poincaré group $ISO(1,3)$ can be introduced to describe the gravitational field as a gauge field of the external rotations and translations.
- Not only an energy-momentum tensor of matter arises, but also a nontrivial spin density tensor which operates as source of torsion $\Longrightarrow$ an extended correspondence between the geometry of the space-time and the properties of matter.

Gauge formalism of metric-affine geometry

- The general case does not assume anything so one has a manifold with curvature, torsion and nonmetricity.

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- Affine group $A(4,\mathcal{R}) = R^4 \otimes GL(4,R)$ is the semiproduct of the translation group R^4 and the general linear group $GL(4,R)$. gauge connection with an independent local metric structure[1]:

$$A_\mu = e^a{}_\mu P_a + \tilde{\omega}^a{}_{b\mu} L_a{}^b \,,$$
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- Generators of the group $A(4,\mathcal{R})$:

$$[P_a, P_b] = 0\,,$$
$$\left[L_a{}^b, P_c\right] = i\,\delta^b{}_c\,P_a\,,$$
$$\left[L_a{}^b, L_c{}^d\right] = i\left(\delta^b{}_c\,L_a{}^d - \delta_a{}^d\,L_c{}^b\right)\,.$$

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\begin{aligned}
G_{ab\mu} &= \partial_\mu g_{ab} - g_{ac}\,\tilde{\omega}^c{}_{b\mu} - g_{bc}\,\tilde{\omega}^c{}_{a\mu}\,, \\
F^a{}_{\mu\nu} &= \partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu + \tilde{\omega}^a{}_{b\mu}\,e^b{}_\nu - \tilde{\omega}^a{}_{b\nu}\,e^b{}_\mu\,, \\
F^a{}_{b\mu\nu} &= \partial_\mu \tilde{\omega}^a{}_{b\nu} - \partial_\nu \tilde{\omega}^a{}_{b\mu} + \tilde{\omega}^a{}_{c\mu}\,\tilde{\omega}^c{}_{b\,\nu} - \tilde{\omega}^a{}_{c\nu}\,\tilde{\omega}^c{}_{b\mu}\,.
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- Correspondence with the curvature, torsion and nonmetricity tensors:

$$\begin{aligned} G_{ab\mu} &= g_{ac} g_{bd} e^{c\lambda} e^{d\rho} Q_{\mu\lambda\rho}, \\ F^a{}_{\mu\nu} &= e^a{}_\lambda T^\lambda{}_{\nu\mu}\,, \\ F^a{}_{b\mu\nu} &= g_{bc}\,e^a{}_\lambda e^{c\rho} \tilde{R}^\lambda{}_{\rho\mu\nu}\,. \end{aligned}$$

Dynamics of metric-affine geometry

- Gravitational action with dynamical torsion and nonmetricity:

$$S = \int d^4x \sqrt{-g} \left[\mathcal{L}_m - \frac{1}{16\pi} \mathcal{L}_g(\tilde{\mathcal{R}}, \mathcal{T}, \mathcal{Q}) \right] . \qquad (3)$$

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$$\frac{\delta S_g}{\delta e^a{}_\nu} = 16\pi\theta_a{}^\nu , \quad (4)$$

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Here $\theta_a{}^\nu$ is the energy-momentum tensor (canonical) and $\Delta_a{}^{b\nu}$ is the hypermomentum density tensor.

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Here $\theta_a{}^\nu$ is the energy-momentum tensor (canonical) and $\Delta_a{}^{b\nu}$ is the hypermomentum density tensor.

- $GL(4, R)$ group allows the definition of a large number of scalar invariants depending on the aforementioned tensors.

- General quadratic gravitational action with dynamical torsion and nonmetricity:

$$
\begin{aligned}
S = \int d^4x\sqrt{-g}\Big\{\mathcal{L}_m + \frac{1}{16\pi}\Big[& -\tilde{R} + a_1\tilde{R}^2 + a_2\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\lambda\rho\mu\nu} + a_3\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\rho\lambda\mu\nu} \\
& + a_4\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\mu\nu\lambda\rho} + a_5\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\lambda\mu\rho\nu} + a_6\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\mu\lambda\rho\nu} + a_7\tilde{R}_{\rho\lambda\mu\nu}\tilde{R}^{\mu\lambda\rho\nu} \\
& + a_8\tilde{R}_{\mu\nu}\tilde{R}^{\mu\nu} + a_9\tilde{R}_{\mu\nu}\tilde{R}^{\nu\mu} + a_{10}\hat{R}_{\mu\nu}\hat{R}^{\mu\nu} + a_{11}\hat{R}_{\mu\nu}\hat{R}^{\nu\mu} + a_{12}\tilde{R}_{\mu\nu}\hat{R}^{\mu\nu} \\
& + a_{13}\tilde{R}_{\mu\nu}\hat{R}^{\nu\mu} + a_{14}\tilde{R}^{\lambda}{}_{\lambda\mu\nu}\tilde{R}^{\rho}{}_{\rho}{}^{\mu\nu} + a_{15}\tilde{R}^{\lambda}{}_{\lambda\mu\nu}\tilde{R}^{\mu\nu} + a_{16}\tilde{R}^{\lambda}{}_{\lambda\mu\nu}\hat{R}^{\mu\nu} \\
& + b_1 T_{\lambda\mu\nu}T^{\lambda\mu\nu} + b_2 T_{\lambda\mu\nu}T^{\mu\lambda\nu} + b_3 T^{\lambda}{}_{\lambda\nu}T^{\mu}{}_{\mu}{}^{\nu} + c_1 T_{\lambda\mu\nu}Q^{\mu\lambda\nu} \\
& + c_2 T^{\lambda}{}_{\lambda\nu}Q^{\nu\mu}{}_{\mu} + c_3 T^{\lambda}{}_{\lambda\nu}Q^{\mu\nu}{}_{\mu} + d_1 Q_{\lambda\mu\nu}Q^{\lambda\mu\nu} + d_2 Q_{\lambda\mu\nu}Q^{\mu\lambda\nu} \\
& + d_3 Q^{\lambda}{}_{\lambda\nu}Q^{\mu}{}_{\mu}{}^{\nu} + d_4 Q_{\nu}{}^{\lambda}{}_{\lambda}Q^{\nu\mu}{}_{\mu} + d_5 Q^{\lambda}{}_{\lambda\nu}Q^{\nu\mu}{}_{\mu}\Big]\Big\}\,. \qquad (6)
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Overview of the Talk

MAG models with dynamical torsion and nonmetricity

- In order to have a theory such that when $T = Q = 0$ one recovers GR, one can relate the constants.

[2] S. Bahamonde and J. G. Valcarcel, JCAP **09**, 057 (2020).

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MAG models with dynamical torsion and nonmetricity

- In order to have a theory such that when $T = Q = 0$ one recovers GR, one can relate the constants.
- Quadratic gravitational action with dynamical torsion and nonmetricity in **Weyl-Cartan geometry (**$Q_{\lambda\mu\nu} = g_{\mu\nu} W_\lambda$**)**

$$S = \int d^4x\sqrt{-g}\Big\{\mathcal{L}_{\rm m} + \frac{1}{64\pi}\Big[-4R - 6d_1\tilde{R}_{\lambda[\rho\mu\nu]}\tilde{R}^{\lambda[\rho\mu\nu]} - 9d_1\tilde{R}_{\lambda[\rho\mu\nu]}\tilde{R}^{\mu[\lambda\nu\rho]} + 8\,d_1\tilde{R}_{[\mu\nu]}\tilde{R}^{[\mu\nu]} + \frac{1}{8}(32e_1 + 8e_2 + 17d_1)\,\tilde{R}^{\lambda}{}_{\lambda\mu\nu}\tilde{R}^{\rho}{}_{\rho}{}^{\mu\nu} - 7d_1\tilde{R}_{[\mu\nu]}\tilde{R}^{\lambda}{}_{\lambda}{}^{\mu\nu} + 3\,(1-2a_2)\,T_{[\lambda\mu\nu]}T^{[\lambda\mu\nu]}\Big]\Big\}. \tag{7}$$

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- Absence of a general Birkhoff's theorem in MAG: new spherically and axially symmetric vacuum solutions with independent dynamical torsion and nonmetricity fields[2,3]

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Spherical symmetry

- Metric, torsion and nonmetricity in spherically symmetric space-times ($\#2 + \#8 + \#2 = \#12$):

$$\mathcal{L}_\xi g_{\mu\nu} = \mathcal{L}_\xi T^\lambda{}_{\mu\nu} = \mathcal{L}_\xi W_\mu = 0 \implies \mathcal{L}_\xi \tilde{R}_{\lambda\rho\mu\nu} = 0$$

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- By solving these equations we find that torsion and nonmetricity behave as

$$T^t{}_{tr} = a(r)\,, \quad T^r{}_{tr} = b(r)\,, \quad T^{\theta_k}{}_{t\theta_k} = f(r)\,, \quad T^{\theta_k}{}_{r\theta_k} = g(r)$$
$$T^{\theta_k}{}_{t\theta_l} = e^{a\theta_k}\, e^b{}_{\theta_l}\, \epsilon_{ab}\, d(r)\,, \quad T^{\theta_k}{}_{r\theta_l} = e^{a\theta_k}\, e^b{}_{\theta_l}\, \epsilon_{ab}\, h(r)\,,$$
$$T^t{}_{\theta_k\theta_l} = \epsilon_{kl}\, k(r)\, \sin\theta_1\,, \quad T^r{}_{\theta_k\theta_l} = \epsilon_{kl}\, l(r)\, \sin\theta_1\,,$$
$$W_\lambda = (w_1(r), w_2(r), 0, 0)\,,$$

whereas the metric is in the standard spherically symmetric form:

$$ds^2 = \Psi_1(r)\, dt^2 - \frac{dr^2}{\Psi_2(r)} - r^2 \left(d\theta_1^2 + \sin^2\theta_1 d\theta_2^2\right)\,.$$

Here, ϵ_{kl} is the Levi-Civita symbol in two dimensions.

Spherical symmetry - Solving the field equations

The field eqs are very involved. To solve them we use the following strategy:

1. **Imposing regularity:** In general, the solutions can have a singular behaviour. To ensure regularity, one can analyse the torsion/nonmetricity tensors referred to the rotated basis $\vartheta^a = \Lambda^a{}_b e^b$.

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$$
\begin{aligned}
b(r) &= a(r)\sqrt{\Psi_1(r)\Psi_2(r)}\,, \qquad & f(r) &= -\,g(r)\sqrt{\Psi_1(r)\Psi_2(r)}\,,\\
d(r) &= -\,h(r)\sqrt{\Psi_1(r)\Psi_2(r)}\,, & l(r) &= k(r)\sqrt{\Psi_1(r)\Psi_2(r)}\,,\\
w_1(r) &= -\,w_2(r)\sqrt{\Psi_1(r)\Psi_2(r)}\,.
\end{aligned}
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Spherical symmetry - Solving the field equations

2. **Solve the weak field limit:** The weak field limit of the field equations become

$$\nabla_\rho \nabla_\lambda T^{\lambda\rho}{}_{\mu} + \nabla_\rho \nabla^\rho T^{\lambda}{}_{\mu\lambda} - \nabla_\rho \nabla_\mu T^{\lambda\rho}{}_{\lambda} = 0\,,$$
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These equations can be solved, yielding

$$w_1(r) = -\,\kappa_d \int \sqrt{\frac{\Psi_1(r)}{\Psi_2(r)}}\,\frac{dr}{r^2}\,,$$

$$b(r) = r f'(r) + f(r) + \frac{\kappa_d}{2r}\sqrt{\frac{\Psi_1(r)}{\Psi_2(r)}}\,,$$

where κ_d is an integration constant which represents the dilaton charge.

Spherical symmetry - Solving the field equations

1. The final solution for the metric behaves as Reissner-Nordström

$$g_{tt} = -1/g_{rr} \equiv \Psi(r) = 1 - \frac{2m}{r} + \frac{d_1\kappa_{\rm s}^2 - 4e_1\kappa_{\rm d,e}^2}{r^2}\,. \tag{8}$$

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2. Nonmetricity sector:

$$W_\mu = \frac{\kappa_{\rm d,e}}{r}\,(1, -1/\Psi(r), 0, 0)\;. \tag{9}$$

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$$W_\mu = \frac{\kappa_{\rm d,e}}{r} \left(1, -1/\Psi(r), 0, 0\right) \,. \tag{9}$$

3. Torsion sector:

$$\bar{\mathcal{S}}^a = -\frac{\kappa_{\rm s}}{r}(1,1,0,0) \,, \tag{10}$$

$$\bar{\mathcal{T}}_2^{abc} = \frac{\kappa_{\rm s}}{3r} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} \,. \tag{11}$$

Dilation and spin charges

What do κ_s (dilation charge) and $\kappa_{d,e}$ (spin charge) physically represent?

Point 1 - Hypermomentum density

In the geometric scheme of MAG, not only an energy-momentum tensor of matter arises as source of curvature, but also a hypermomentum density tensor which operates as source of torsion and nonmetricity.

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Point 2 - Dilation and spin charges

In Weyl-Cartan geometry, hypermomentum density tensor splits into spin and dilation currents, which carry their own charges and provide a RN solution.

Dilation and spin charges

When these charges might be important?

Significant effects are contemplated only around **extreme gravitational systems**, such as **neutron stars** with intense magnetic fields and sufficiently oriented elementary spins or **black holes** endowed with spin and dilation charges.

Particle motion in MAG

- The equations of motion of test bodies with microstructure coupled to the torsion and nonmetricity tensors become[4]

$$\dot{p}^{\mu} + \Gamma^{\mu}{}_{\lambda\rho}\, p^{\lambda} u^{\rho} + N_{[\lambda\rho]}{}^{\mu} p^{\rho} u^{\lambda} + \tilde{R}_{\lambda\rho\sigma}{}^{\mu} \triangle^{\rho\lambda}\, u^{\sigma} = 0\,.$$

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- This eq. reduces to the standard geodesic one ($\dot{p}^{\mu} + \Gamma^{\mu}{}_{\lambda\rho}\, p^{\lambda} u^{\rho} = 0$) when the hypermomentum of the test body vanishes and also when the particle are bosons.

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- Using the standard geodesic approach, we find

$$\frac{1}{2}\dot{r}^2 + V(r) = 0\,, \quad V(r) = -\frac{1}{2}c^2 E^2 + \frac{1}{2}\Psi(r)\left(\frac{J^2}{r^2} + \sigma c^2\right)\,,$$

where E an J are the conserved charges and $\sigma = 0(\sigma = 1)$ represents massless(massive) particles.

[4] D. Puetzfeld and Y. N. Obukhov, Phys. Rev. D **76** (2007), 084025.

Observational constrains

- Let us now consider the case where the effect of torsion dominates over the contribution of nonmetricity.

[5] S. Bahamonde and J. Gigante Valcarcel, Eur. Phys. J. C **81** (2021) no.6, 495.

Observational constrains

- Let us now consider the case where the effect of torsion dominates over the contribution of nonmetricity.
- Indeed, due to the presence of a magnetic field in white dwarfs, it is expected that Sirius B can have sufficiently oriented elementary spins in comparison with an effective dilation charge, therefore, $\kappa_{s,\mathrm{SiriusB}} \gg \kappa_{d,\mathrm{SiriusB}}$.

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- **Perihelion shift+ Gravitational redshift:** Assuming the same approximation in Sgr A* and considering the universality of the coupling constant d_1, we find[5]

$$1.396 \cdot 10^{10} \leq \frac{\kappa_{s,\mathrm{SgrA*}}}{\kappa_{s,\mathrm{SiriusB}}} \leq 1.688 \cdot 10^{10}\,.$$

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- To the best of our knowledge, this bound provides the first observational comparison between the spin charges of a supermassive black hole and a degenerate star.

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Extension to axisymmetric space-times

- Metric, torsion and nonmetricity tensors in symmetric space-times:

$$\mathcal{L}_\xi g_{\mu\nu} = \mathcal{L}_\xi T^\lambda{}_{\mu\nu} = \mathcal{L}_\xi Q^\lambda{}_{\mu\nu} = 0 \implies \mathcal{L}_\xi \tilde{R}^\lambda{}_{\rho\mu\nu} = 0\,.$$

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- Stationary and axisymmetric space-times:

$$\#10 \to \#4 \begin{cases} ds^2 = \Psi_1(r,\vartheta)\,dt^2 - \frac{dr^2}{\Psi_2(r,\vartheta)} \\ -r^2\Psi_3(r,\vartheta)\Big[d\vartheta^2 + \sin^2\vartheta(d\varphi - \Psi_4(r,\vartheta)dt)^2\Big] \end{cases} ;$$

$$\#24 \Big\{ T^{\lambda}{}_{\mu\nu} = T^{\lambda}{}_{\mu\nu}(r,\vartheta)$$

$$\#4 \Big\{ W_\mu = (W_t(r,\vartheta), W_r(r,\vartheta), W_\vartheta(r,\vartheta), W_\varphi(r,\vartheta))\,.$$

Axisymmetric space-times - Kerr-Newmann de-Sitter

- Rotating Kerr-Newman metric structure[6]:

$$ds^2 = \Psi(r,\vartheta)\,dt^2 - \frac{r^2 + a^2\cos^2\vartheta}{(r^2 + a^2\cos^2\vartheta)\,\Psi(r,\vartheta) + a^2\sin^2\vartheta}\,dr^2$$
$$- \left(r^2 + a^2\cos^2\vartheta\right)d\vartheta^2 + 2a\left(1 - \Psi(r,\vartheta)\right)\sin^2\vartheta\,dtd\varphi$$
$$- \sin^2\vartheta\left[r^2 + a^2 + a^2\left(1 - \Psi(r,\vartheta)\right)\sin^2\vartheta\right]d\varphi^2\,, \tag{12}$$

$$\Psi(r,\vartheta) = 1 - \frac{\left[2mr + 4e_1(\kappa^2_{\mathrm{d,e}} + \kappa^2_{\mathrm{d,m}}) - d_1\kappa^2_{\mathrm{s}}\right]}{r^2 + a^2\cos^2\vartheta}\,. \tag{13}$$

[6] S. Bahamonde and J. G. Valcarcel, JCAP **01** (2022) no.01, 011.

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- Field strength tensors:

$$\bar{R}_{[\mu\nu]} = \frac{1}{12}\varepsilon^{\lambda}{}_{\sigma\mu\nu}\nabla_\lambda\bar{S}^\sigma + \frac{1}{2}\nabla_\lambda\bar{t}^{\lambda}{}_{\mu\nu}\,; \quad \tilde{R}^{\lambda}{}_{\lambda\mu\nu} = 4\nabla_{[\nu}W_{\mu]}\,;$$
$$\bar{R}^{\lambda}{}_{[\mu\nu\rho]} = \frac{1}{6}\varepsilon^{\lambda}{}_{\sigma[\rho\nu}\nabla_{\mu]}\bar{S}^\sigma + \nabla_{[\mu}\bar{t}^{\lambda}{}_{\rho\nu]} + \frac{1}{4}\varepsilon^{\lambda}{}_{\tilde{\omega}\sigma[\rho}\mathring{t}_1^{\sigma}{}_{\mu\nu]}\bar{S}^{\tilde{\omega}}$$
$$- \frac{1}{18}\varepsilon_{\sigma\mu\nu\rho}\mathring{T}_1^{\lambda}\bar{S}^\sigma\,.$$

[6] S. Bahamonde and J. G. Valcarcel, JCAP **01** (2022) no.01, 011.

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- Nonmetricity sector:(no approx.)

$$w_1(r,\vartheta) = \frac{\kappa_{\rm d,e} r - a\,\kappa_{\rm d,m}\cos\vartheta}{r^2 + a^2\cos^2\vartheta}\,, \quad w_3(r,\vartheta) = 0\,,$$
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- Torsion sector (decoupling limit between the spin and the orbital angular momentum $|a\kappa_{\rm s}| \ll 1$):

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Gravitational spin-orbit interaction

- We found a solution in the decoupling limit $a\kappa_s \ll 1$, which ensures that the Maxwell equation and closure conditions are fulfilled by the field strength tensors of torsion

$$\nabla_\lambda \tilde{R}^\lambda{}_{[\rho\mu\nu]} \quad = \quad \nabla_\mu \tilde{R}^{[\mu\nu]} = 0\,, \quad \nabla_{[\sigma}\tilde{R}_{\lambda[\rho\mu\nu]]} = \nabla_{[\lambda}\tilde{R}_{[\mu\nu]]} = 0\,.$$

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The dynamics of torsion and nonmetricity alters the geometry of the space-time $\Longrightarrow$

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- **Gravitational spin-orbit interaction:**

$$\mathcal{H}_{\rm LS} = \frac{1}{m_{\rm e}^2 r}\frac{\partial V}{\partial r}\mathbf{L}\cdot\mathbf{S} \approx \frac{d_1}{2r}\frac{\partial g_{tt}}{\partial r} a\kappa_{\rm s}\cos\vartheta \qquad (14)$$

- It is well known that the most general axisymmetric system in vacuum that can describe a BH type D in **GR** contains[7]:

Mass	M
Angular momentum	a
Taub-NUT charge	l
Acceleration	α

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- Further, one can add a cosmological constant Λ and a electric charge q_e and magnetic charge q_m.
- The solution in GR is called Plebanski-Damianski solution.

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- The Plebanski-Damianski metric was recently presented in an improved form with $\Lambda = 0$ in by Podolský and Vrátný (Phys. Rev. D **104** (2021), 084078), and it can be written as

$$ds^2 = \Omega^{-2}(r,\vartheta)\Big\{\Phi_1(r,\vartheta)\left[dt - \big(a\sin^2\vartheta + 2l(\chi - \cos\vartheta)\big)\,d\varphi\right]^2 - \frac{dr^2}{\Phi_1(r,\vartheta)} - \frac{d\vartheta^2}{\Phi_2(r,\vartheta)} - \Phi_2(r,\vartheta)\sin^2\vartheta\big[a\,dt - \big(r^2 + a^2 + l^2 + 2\chi a l\big)\,d\varphi\big]^2\Big\}\,.$$

where Φ_i, Ω are cumbersome functions depending on these parameters.

[8] S. Bahamonde, J. G. Valcarcel and L. Järv, JCAP **04** (2022) no.04, 011.

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  where Φ_i, Ω are cumbersome functions depending on these parameters.

- We just found this new form with the cosmological constant[8] with $\Phi_1(r,\vartheta) = \frac{Q(r)}{\rho^2(r,\vartheta)}\,, \quad \Phi_2(r,\vartheta) = \frac{P(\vartheta)}{\rho^2(r,\vartheta)}$, and $\rho^2(r,\vartheta) = r^2 + (a\cos\vartheta + l)^2$. Here, $Q(r), \Omega(\vartheta)$ include the PD quantities.

[8] S. Bahamonde, J. G. Valcarcel and L. Järv, JCAP **04** (2022) no.04, 011.

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$$w_3(r,\vartheta) = -\,\kappa_{\mathrm{d,m}}\sqrt{K(\vartheta) - \left(\frac{\cot\vartheta - \gamma\csc\vartheta}{P(\vartheta)}\right)^2}\,,$$

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$$\bar{T}^{\vartheta}{}_{\varphi t} = -\,\bar{T}^{\varphi}{}_{\vartheta t}\sin^2\vartheta = -\,\bar{T}^{\vartheta}{}_{\varphi r}\frac{Q(r)}{\rho^2(r,\vartheta)} = \bar{T}^{\varphi}{}_{\vartheta r}\frac{Q(r)}{\rho^2(r,\vartheta)}\sin^2\vartheta = \frac{\kappa_s\sin\vartheta}{r} + \mathcal{O}(x_i\kappa_s)\,.$$

- Similarly as electromagnetism, the torsion behaves as a Coulomb-like quantity depending on a spin charge κ_s and the non-metricity on the dilation charge κ_d.

Extension with traceless part of nonmetricity - ongoing

- All previous works assumed Weyl-Cartan gravity (torsion+curvature+non-metricity only with Weyl part).

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- The general MAG contains the complete $Q_{\alpha\beta\gamma}$ which can be decomposed as

$$Q_{\lambda\mu\nu} = \overbrace{g_{\mu\nu}W_{\lambda}}^{\textbf{Weyl part}} + \overbrace{\not{Q}_{\lambda\mu\nu}}^{\textbf{Traceless part}} \,, \tag{15}$$

where

$$\not{Q}_{\lambda\mu\nu} = g_{\lambda(\mu}\Lambda_{\nu)} - \frac{1}{4}g_{\mu\nu}\Lambda_{\lambda} + \frac{1}{3}\varepsilon_{\lambda\rho\sigma(\mu}\Omega_{\nu)}{}^{\rho\sigma} + q_{\lambda\mu\nu} \,,$$

where $\varepsilon_{\lambda\rho\mu\nu}$ is the Levi-Civita (density) tensor

- Further, we introduced:

$$\Lambda_\mu = \frac{4}{9}\left(Q^{\nu}{}_{\mu\nu} - W_\mu\right) ,$$

$$\Omega_\lambda{}^{\mu\nu} = -\left[\varepsilon^{\mu\nu\rho\sigma} Q_{\rho\sigma\lambda} + \varepsilon^{\mu\nu\rho}{}_{\lambda}\left(\frac{3}{4}\Lambda_\rho - W_\rho\right)\right] ,$$

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  which constitute a vector, and two traceless and pseudotraceless tensors.
- We are now finishing a paper where we found the first exact spherically symmetric black hole solution with shears, having the form of the metric

$$\Psi(r) = 1 - \frac{2m}{r} + \frac{d_1\kappa_{\rm s}^2 - 4e_1\kappa_{\rm d}^2 - 2f_1\kappa_{\rm sh}^2}{r^2} , \qquad (16)$$

  where $\kappa_{\rm s}$ is the spin charge (torsion), $\kappa_{\rm d}$ is the dilation charge (Weyl part of nonmetricity) and $\kappa_{\rm sh}$ the shear charge (traceless part of nonmetricity).

Conclusions - Messages to home

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- There are three alternative ways of representing gravity as GR and they are indistinguishable at the classical level (GR, TEGR, STEGR).
- The MAG are gauge theories of gravity with the field strength tensors given by the curvature, torsion and nonmetricity.

Conclusions

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- In the 2nd paper, we studied the phenomenology of particle and provided observational constrains for the charges.
- In the 3rd paper, we obtained an axially symmetric solution behaving as a Kerr-Newman-de Sitter solution (in the decoupling limit).

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- In the 3rd paper, we obtained an axially symmetric solution behaving as a Kerr-Newman-de Sitter solution (in the decoupling limit).
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Conclusions

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- Future: search of a gravitational spin-orbit interaction in MAG beyond the Kerr-Newman space-time (MAG is the main candidate to describe a spin-orbit interaction beyond GR).