Is really General Relativity the correct starting point for quantum gravity?

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1. Metric-Affine theories of gravity

- In metric-affine theories, both the **metric** $g_{\mu\nu}$ and the **connection** $\Gamma^{\alpha}{}_{\mu\nu}$ are the dynamical variables and *a priori* they **do not depend** on each other.
- One can define the failure of the connection to be metric by the so-called **non-metricity tensor** given by
$$Q_{\alpha\mu\nu} \equiv \nabla_{\alpha} g_{\mu\nu}\,. \tag{1}$$
In addition, one can define the antisymmetric part of this connection with the **torsion tensor**:
$$T^{\alpha}{}_{\mu\nu} \equiv 2\Gamma^{\alpha}{}_{[\mu\nu]}. \tag{2}$$
The unique connection which is metric compatible and is symmetric is the **Levi-Civita** connection which is based on the Christoffel symbols,
$$\left\{ {\alpha \atop \mu\nu} \right\} = \frac{1}{2} g^{\alpha\lambda}(g_{\lambda\nu,\mu} + g_{\mu\lambda,\nu} - g_{\mu\nu,\lambda})\,. \tag{3}$$
- A general connection $\Gamma^{\alpha}{}_{\mu\nu}$ can be decomposed into three different pieces:
$$\Gamma^{\alpha}{}_{\mu\nu} = \left\{ {\alpha \atop \mu\nu} \right\} + K^{\alpha}{}_{\mu\nu} + L^{\alpha}{}_{\mu\nu}\,, \tag{4}$$
where $K^{\alpha}{}_{\mu\nu} = \frac{1}{2}T^{\alpha}{}_{\mu\nu} + T_{(\mu}{}^{\alpha}{}_{\nu)}$ depend on torsion and it is called the **contortion tensor** and $L^{\alpha}{}_{\mu\nu} = \frac{1}{2}Q^{\alpha}{}_{\mu\nu} - Q_{(\mu}{}^{\alpha}{}_{\nu)}$ depend on the non-metricity tensor and is called **disformation tensor**.
- The **curvature tensor** can be then defined for a general connection $\Gamma^{\alpha}{}_{\mu\nu}$ as
$$R^{\alpha}{}_{\beta\mu\nu}(\Gamma) = \partial_{\mu}\Gamma^{\alpha}{}_{\nu\beta} - \partial_{\nu}\Gamma^{\alpha}{}_{\mu\beta} + \Gamma^{\alpha}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\beta} - \Gamma^{\alpha}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\mu\beta}\,, \tag{5}$$
and then corresponding Ricci tensor and Ricci scalar are constructed from it.

2. What do curvature, torsion and non-metricity represent geometrically?

- **Curvature** $R^{\alpha}{}_{\beta\mu\nu}$: rotation experienced by a vector when it is parallel transported along a closed curve
- **Torsion** $T^{\alpha}{}_{\mu\nu}$: non-closure of the parallelogram formed when two infinitesimal vectors are parallel transported along each other.
- **Non-metricity** $Q_{\alpha\mu\nu}$: measures how much the length and angle of vectors change as we parallel transport them, so in metric spaces the length of vectors is conserve

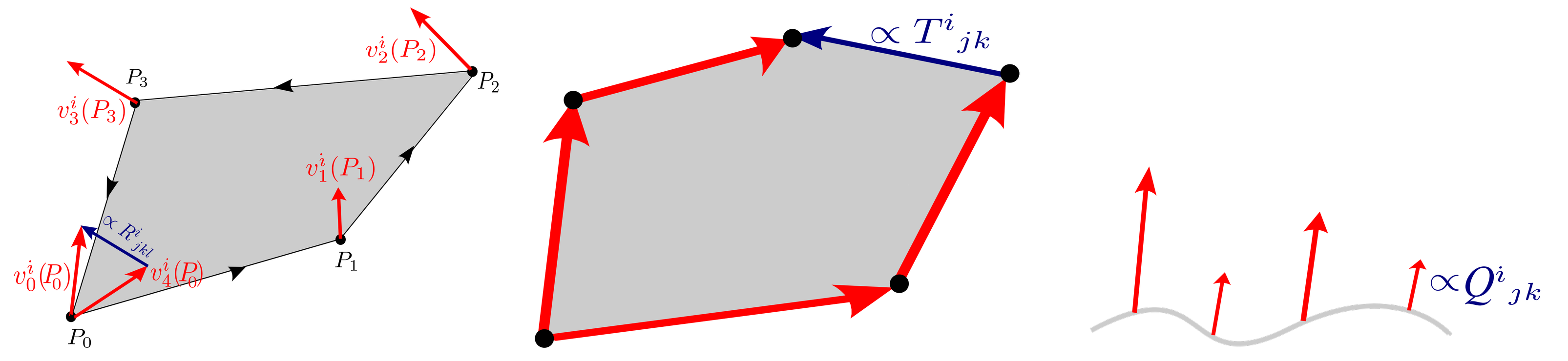

Figure: Geometrical representation of curvature, torsion and non-metricity for a 3-dimensional space

3. Trinity of gravity I: General Relativity (GR), Teleparallel equivalent of GR and Coincident GR

General Relativity (GR):

- GR is the geometrical theory of gravity which assumes a manifold with **non-vanishing curvature** $R^{\alpha}{}_{\beta\mu\nu} \neq 0$ whilst both the **non-metricity tensor** and the **torsion tensor are zero**: $Q_{\alpha\mu\nu} = \nabla_{\alpha} g_{\mu\nu} = 0,\ T^{\alpha}{}_{\mu\nu} = 0$.
- According to (4), the connection is equal to $\mathring{\Gamma}^{\alpha}{}_{\mu\nu} = \left\{ {\alpha \atop \mu\nu} \right\}$, which is the standard Levi-Civita connection, and the contraction of the curvature $\mathcal{R}^{\alpha}{}_{\beta\alpha\nu}$ becomes symmetric.
- The **action of this theory** is constructed from the contraction of the curvature tensor with the metric, known as the Ricci scalar $\mathcal{R} = \mathcal{R}^{\alpha}{}_{\beta\alpha\nu}g^{\beta\nu}$, and reads
$$\mathcal{S}_{\rm GR} = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\,\mathcal{R}(g)\,. \tag{6}$$
Here, $\kappa^2 = 8\pi G$ and this action only depends on the metric, hence, by varying this action with respect to the metric, one gets the standard **Einstein field equations**.
- Matter fields follow the geodesic equation.

4. Trinity of gravity II: General Relativity (GR), Teleparallel equivalent of GR and Coincident GR

Teleparallel equivalent of General Relativity (TEGR):

- TEGR is an alternative theory which assumes the Weitzenböck connection which has a **non-vanishing torsion, zero curvature** $R^{\alpha}{}_{\beta\mu\nu} = 0$ (flat spacetime) and that the metric **satisfies the compatibility condition** $Q_{\alpha\mu\nu} = \nabla_{\alpha}g_{\mu\nu} = 0$.
- The fundamental variables for this theory are the **tetrads** $h^{a}{}_{\nu}$ and the **purely inertial spin connection** $\Gamma^{\alpha}{}_{\mu\nu} = (\Lambda^{-1})^{\alpha}{}_{\lambda}\partial_{\mu}\Lambda^{\lambda}{}_{\nu}$ with Λ being the Lorentz matrix.
- One can then consider **contractions of torsion** to get an alternative theory of gravity. Since torsion is antisymmetric one can only have **three independent contractions** of it
$$\mathbb{T} = c_1 T^{\rho}{}_{\mu\nu}T_{\rho}{}^{\mu\nu} + c_2 T^{\rho}{}_{\mu\nu}T^{\nu\mu}{}_{\rho} + c_3 T^{\lambda}{}_{\lambda\mu}T_{\nu}{}^{\nu\mu}\,, \tag{7}$$
with c_i being coupling constants. The theory constructed from $\mathbb{T}$ is called **New General Relativity**.
- If one assumes zero curvature and zero non-metricity in Eq. (5), and then computes the Ricci scalar, one gets
$$R = \mathcal{R} + \mathring{\mathbb{T}} - \frac{2}{\sqrt{-g}}\partial_{\mu}\left(\sqrt{-g}T^{\sigma}{}_{\sigma}{}^{\mu}\right) = 0 \quad\Rightarrow\quad \mathcal{R} = -\mathring{\mathbb{T}} + 2\mathcal{D}_{\mu}\left(T^{\sigma}{}_{\sigma}{}^{\mu}\right) := -\mathring{\mathbb{T}} + B\,, \tag{8}$$
where $\mathcal{R}$ is the curvature computed with the Levi-Civita connection and we have defined the **torsion scalar** as $\mathring{\mathbb{T}} = \mathbb{T}$ with the coefficient constants $c_1 = \frac{1}{4},\ c_2 = \frac{1}{2},\ c_3 = -1$.
- The **TEGR action** is defined from the torsion scalar as follows
$$\mathcal{S}_{\rm TEGR} = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\,\mathring{\mathbb{T}}(h,\Lambda)\,, \tag{9}$$
and then, since the Einstein-Hilbert action (6) **differs only by a boundary term** B with respect to the above action (see (8)), by varying the TEGR action with respect to the tetrads, one also gets the Einstein field equations. Then, TEGR is **classically equivalent to GR at the level of the field equations**.

5. Trinity of gravity III: General Relativity (GR), Teleparallel equivalent of GR and Coincident GR

Coincident General Relativity (CGR):

- CGR in not a geometrical theory of gravity since assumes that both the **curvature and torsion tensor are zero** $R^{\alpha}{}_{\beta\mu\nu} = T^{\rho}{}_{\mu\nu} = 0$. In this theory, the **non-metricity condition is not satisfied** $Q_{\alpha\mu\nu} = \nabla_{\alpha}g_{\mu\nu} \neq 0$, therefore the **non-metricity tensor** plays the role of being the field strength.
- Similarly as in TEGR, one can then construct theories with the **contractions of the non-metricity tensor**. There are now five possible non-trivial contractions that one can construct
$$\mathcal{Q} = \frac{c_1}{4}Q_{\alpha\beta\gamma}Q^{\alpha\beta\gamma} - \frac{c_2}{2}Q_{\alpha\beta\gamma}Q^{\beta\alpha\gamma} - \frac{c_3}{4}Q_{\alpha}Q^{\alpha} + (c_4-1)\tilde{Q}_{\alpha}\tilde{Q}^{\alpha} + \frac{c_5}{2}Q_{\alpha}\tilde{Q}^{\alpha}\,, \tag{10}$$
with c_i being coupling constants, $Q_{\alpha} = Q_{\alpha\lambda}{}^{\lambda}$ and $\tilde{Q}_{\alpha} = Q^{\lambda}{}_{\lambda\alpha}$.
- By setting that both curvature and torsion are equal to zero in (5) and then doing the correct contractions, one finds that the Ricci scalar computed with the Levi-Civita connection $\mathcal{R}$ is related to a specific scalar constructed from the non-metricity tensor, namely
$$R = \mathcal{R} + \mathring{\mathcal{Q}} + \mathcal{D}_{\alpha}(Q^{\alpha} - \tilde{Q}^{\alpha}) = 0 \rightarrow \mathcal{R} = -\mathring{\mathcal{Q}} + \mathcal{D}_{\alpha}(\tilde{Q}^{\alpha} - Q^{\alpha})\,. \tag{11}$$
Here, $\mathring{\mathcal{Q}}$ is the scalar constructed from (10) with $c_i = 1$ ($i = 1,..,5$). One can notice that $\mathring{\mathcal{Q}}$ and $\mathcal{R}$ **differs only by a boundary term**.
- Thus, similarly as before, one can formulate a **classically equivalent theory of gravity to GR** from the following action
$$\mathcal{S}_{\rm CGR} = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\,\mathring{\mathcal{Q}}(g,\xi)\,, \tag{12}$$
since again this action is **equivalent** up to a total derivative term with respect to the Einstein-Hilbert action.
- The flatness condition again restricts the connection to be purely inertial but now since torsion is also zero, the connection is parametrised by a set of functions ξ^{k} such that $\Gamma^{\alpha}{}_{\mu\nu} = \frac{\partial x^{\alpha}}{\partial \xi^{\lambda}}\partial_{\mu}\partial_{\nu}\xi^{\lambda}$. Thus, the connection can be trivialised by a coordinate transformation $\xi^{\alpha} = x^{\alpha}$ which is known as the **coincident gauge**.
- In the coincident gauge, it is possible to rewrite the action (12) containing only **first derivatives of the metric**
$$\mathcal{S}_{\rm CGR} = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\,g^{\mu\nu}\left(\left\{ {\alpha \atop \beta\mu} \right\}\left\{ {\beta \atop \nu\alpha} \right\} - \left\{ {\alpha \atop \beta\alpha} \right\}\left\{ {\beta \atop \mu\nu} \right\}\right)\,, \tag{13}$$
then, leading to a well-posed variational principle **without any GHY boundary terms**.

6. How are the theories related?

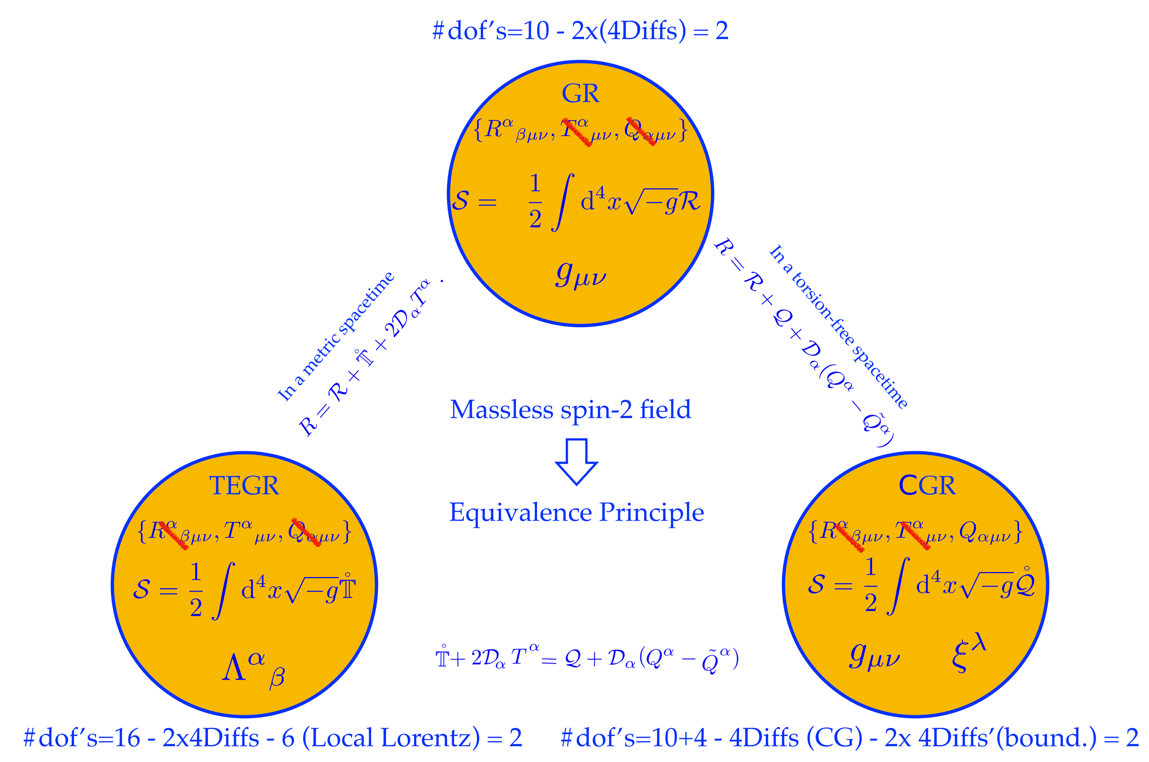

Figure: Trinity of gravity (taken from [1]).

7. Why TEGR or CGR could be good for quantum gravity?

- Since GR, TEGR and CGR have the **same classical field equations**, and all the quantum approaches have been made in GR, it would be interesting to see what happens in quantum approaches in the other two theories.
- Since these three theories are classically equivalent in field equations, it is then a **matter of convention** to say if gravity is the **curvature of a torsionless spacetime**, or **torsion of a curvatureless spacetime**, or if it occurs due to the **non-metricity of a curvatureless and torsionless spacetime**.
- In GR, one needs to introduce the **Gibbons-Hawking-York boundary term**, otherwise the action is not regular. On the other hand in both TEGR and CGR, one **does not need to introduce this boundary term**. Hence, both the TEGR and CGR actions **exhibit a well-posed action principle** and readily admits the **path integral approach to quantization**. As an example, the computation of the **Schwarzschild black hole entropy** in each theory is:
$$\mathcal{S}_{GR} = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\,\mathcal{R} + \int_{\partial M} d^3x\mathcal{K} + \int_{\partial M} d^3x\mathcal{K}_0 = 0 + \infty + \infty = 4\pi GM^2\,, \tag{14}$$
$$\mathcal{S}_{TEGR} = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\,\mathring{\mathbb{T}} = -\frac{1}{2\kappa^2}\int_{\partial M}\sqrt{-g}B = 4\pi GM^2\,, \tag{15}$$
$$\mathcal{S}_{CGR} = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\,\mathring{\mathcal{Q}}\,|_{\xi^{\alpha}=x^{\alpha}} = -\frac{1}{2\kappa^2}\int_{\partial M} d^3x\tilde{Q}^{\mu}n_{\mu} = 4\pi GM^2\,. \tag{16}$$
- GR is based on the **equivalence principle** whose strong version establishes the **local** equivalence between gravitation and inertia. The fundamental asset of quantum mechanics, on the other hand, is the **uncertainty principle**, which is essentially **nonlocal**. It seems that both approaches are not consistent. Further, at quantum scales up to now, it has not been discovered yet if the equivalence principle is always valid or not. In TEGR, one **can** formulate the theory **without the equivalence principle**. Hence, it can **comply with universality but remains a consistent theory in its absence**.
- In CGR (in the coincidence gauge) and TEGR (in the Weitzenböck gauge), there is a way to **define the gravitational energy-momentum tensor** (as a tensor locally conserved) and in GR, this is not possible. Moreover, in both CGR and TEGR one can **separate gravity from inertia**.
- CGR can be written as the **canonical framework for a gauge theory** of the Abelian group of **translations** since $[\nabla_{\mu}, \nabla_{\nu}] = 0$. All the other **forces** are written in the **gauge language** so this is a good hint to explore quantum approaches in this framework.

References

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