Solar System Tests in Modified Teleparallel gravity

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Outline

1. Brief Introduction to Teleparallel theories of gravity
2. New perturbed spherically symmetric solutions
   - Theory and spherical symmetry
   - Perturbed spherically symmetric solutions
3. Phenomenology in $f(T,B)$ gravity
   - Potential and Basic ingredients
   - Solar System Tests
4. Conclusions

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Teleparallel equivalent of GR

- **Teleparallel equivalent of GR** (TEGR) is an alternative and equivalent formulation of gravity from GR which uses the **tetrad** formalism.
- **Tetrads** (or vierbein) e^a_μ are the linear basis on the spacetime manifold, and at each point of the spacetime, tetrads gives us basis for vectors on the tangent space.
- **Notation:** Greek letters: space-time indices; Latin letters: tangent space indices. ${E_m}^\mu$ is the inverse of the tetrad.
- Tetrads satisfy the **orthogonality condition**; ${E_m}^\mu {e^n}_\mu = \delta^n_m$ and ${E_m}^\nu {e^m}_\mu = \delta^\nu_\mu$ and the metric can be reconstructed via

$$g_{\mu\nu} = \eta_{ab} {e^a}_\mu {e^b}_\nu \,, \tag{1}$$

where η_{ab} is the Minkowski metric.

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Connection in Teleparallel gravity

- GR assumes zero torsion and non-zero curvature (Levi-Civita connection) whereas TEGR assumes **zero curvature and non-zero torsion.**
- Then, TEGR has a different connection known as "Weitzenböck connection", defined as

Weitzenböck connection

$$\hat{\Gamma}^{\rho}{}_{\mu\nu} = E_a{}^{\rho} D_\mu e^a{}_\nu = E_a{}^{\rho}(\partial_\mu e^a{}_\nu + \omega^a{}_{b\mu} e^b{}_\nu)\,.$$

- Here, $\omega^a{}_{b\mu} = \Lambda^a{}_c \partial_\mu \Lambda_b{}^c$ is the spin connection which is purely gauge - measures inertial effects.

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Torsion tensor

- By using the Weitzenböck connection connection, one can express the torsion tensor as follows

Torsion tensor

$$T^{\rho}{}_{\mu\nu} = \tilde{\Gamma}^{\rho}{}_{\nu\mu} - \tilde{\Gamma}^{\rho}{}_{\mu\nu} \,.$$

- The torsion tensor is generally non-vanishing, and transforms covariantly under both diffeomorphisms and local Lorentz transformations.
- **Remark:** The pure tetrad formalism was the initial framework used for TG, which chooses a specific frame where the spin connection $\omega^{a}{}_{b\mu}$ vanishes. In this formulation, one needs to be careful choosing the correct tetrad which solves the antisymmetric field equations.

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Flat condition in Teleparallel gravity

- The Weitzenböck connection $\tilde{\Gamma}^{\rho}{}_{\nu\mu}$ is related to the Levi-Civita connection $\Gamma^{\rho}{}_{\nu\mu}$ via

Relationship between connections

$$\tilde{\Gamma}^{\rho}{}_{\nu\mu} = \Gamma^{\rho}{}_{\nu\mu} + K^{\rho}{}_{\mu\nu}\,,$$

where $K^{\rho}{}_{\mu\nu} = \frac{1}{2}(T_{\mu}{}^{\rho}{}_{\nu} + T_{\nu}{}^{\rho}{}_{\mu} - T^{\rho}{}_{\mu\nu})$ is the contortion tensor.

- In this connection, it is easy to verify that the spacetime is globally flat:

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Curvature in Teleparallel gravity

$$R^{a}{}_{b\mu\nu}(\omega^{a}{}_{b\mu}) = \partial_{\mu}\omega^{a}{}_{b\nu} - \partial_{\nu}\omega^{a}{}_{b\mu} + \omega^{a}{}_{c\mu}\omega^{c}{}_{b\nu} - \omega^{a}{}_{c\nu}\omega^{c}{}_{b\mu} \equiv 0\,.$$

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Ricci theorem in Teleparallel gravity

- By splitting the curvature tensor and contracting it with the metric $g^{\mu\nu}R^{\lambda}{}_{\mu\lambda\nu} \equiv R$ (Ricci scalar-general one), one gets

Ricci theorem final in TEGR

$$R = \mathring{R} + T - B = 0 \rightarrow \mathring{R} = -T + B\,,$$

where $B = \frac{2}{e}\partial_\mu(eT^\mu)$ is a boundary term in the action (see later) and $T = \frac{1}{4}T^{\rho}{}_{\mu\nu}T_{\rho}{}^{\mu\nu} + \frac{1}{2}T^{\rho}{}_{\mu\nu}T^{\nu\mu}{}_{\rho} - T^{\lambda}{}_{\lambda\mu}T_{\nu}{}^{\nu\mu}$ is the **scalar torsion.**

Important remark here

The Ricci scalar computed from the Levi-Civita connection $\mathring{R}$ differs from the scalar torsion T by a boundary term B.

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Teleparallel equivalent of General Relativity action

- The TEGR action is formulated based on the torsion scalar T, namely

$$S_{\mathrm{TEGR}} = \int \left[T + 2\kappa^2 L_{\mathrm{m}}\right] e\, d^4x\,.$$

where $\kappa^2 = 8\pi G$, $e = \det(e^a_\mu) = \sqrt{-g}$ and L_{m} is any matter Lagrangian.

- T and the scalar curvature $\mathring{R}$ differs by a boundary term B as $\mathring{R} = -T + B$ so:

Equivalence between field equations

The teleparallel field equations are equivalent to the Einstein field equations.

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Two different ways of understanding gravity

Equivalence on their field equations

VERY IMPORTANT POINT: TEGR has the same equations as GR, so **CLASSICALLY** it is impossible to make any observation to distinguish between them.

Validity of TEGR

VERY IMPORTANT POINT: All classical experiments already done, that have confirmed GR, also can be understood as a confirmation of TEGR.

What happens if we modify TEGR?

If we modify the TEGR action, a priori there is no equivalence between modified theories from GR and modified Teleparallel theories.

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$f(T)$ gravity

- In analogy with $f(\mathring{R})$ gravity, one can consider

$f(T)$ gravity action

$$S_{f(T)} = \int f(T) e\, d^4x\,.$$

- The torsion scalar T depends on the first derivatives of the tetrads $\rightarrow$ **Second order theory**:

Not equivalency between $f(\mathring{R})$ and $f(T)$

Field equations of $f(T) \neq$ Field equations of $f(\mathring{R})$

- The reason is the previous equation:

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- If $f(T,B) = f(-T+B) = f(\mathring{R})$, one finds the $f(\mathring{R})$ theory in the context of TEGR.
- If $f(T,B) = f(T)$, one gets $f(T)$ gravity
- Other theories related to the boundary term such as $T + f(B)$ gravity.

*S. Bahamonde, C. G. Böhmer and M. Wright, Phys. Rev. D **92** (2015) no.10, 104042

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Spherical symmetry in $f(T,B)$ gravity

- Let us assume a spherically symmetric spacetime whose metric is

$$ds^2 = \mathcal{A}(r)dt^2 - \mathcal{B}(r)dr^2 - r^2 d\Omega^2\,,$$

where $\mathcal{A}(r)$ and $\mathcal{B}(r)$ are positive functions, which is reproduced by the off-diagonal tetrad

$$e^a{}_\mu = \begin{pmatrix} \sqrt{\mathcal{A}} & 0 & 0 & 0 \\ 0 & \sqrt{\mathcal{B}}\sin(\theta)\cos(\phi) & r\cos(\theta)\cos(\phi) & -r\sin(\theta)\sin(\phi) \\ 0 & \sqrt{\mathcal{B}}\sin(\theta)\sin(\phi) & r\cos(\theta)\sin(\phi) & r\sin(\theta)\cos(\phi) \\ 0 & \sqrt{\mathcal{B}}\cos(\theta) & -r\sin(\theta) & 0 \end{pmatrix}.$$

- This tetrad is compatible with a vanishing spin connection and also both $T = B = 0$ for the Minkowski case ($\mathcal{A} = 1$ and $\mathcal{B} = 1$).

Spherical symmetry in $f(T, B)$ gravity

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$$e^a{}_\mu = \begin{pmatrix} \sqrt{\mathcal{A}} & 0 & 0 & 0 \\ 0 & \sqrt{\mathcal{B}}\sin(\theta)\cos(\phi) & r\cos(\theta)\cos(\phi) & -r\sin(\theta)\sin(\phi) \\ 0 & \sqrt{\mathcal{B}}\sin(\theta)\sin(\phi) & r\cos(\theta)\sin(\phi) & r\sin(\theta)\cos(\phi) \\ 0 & \sqrt{\mathcal{B}}\cos(\theta) & -r\sin(\theta) & 0 \end{pmatrix} .$$

- This tetrad is compatible with a vanishing spin connection and also both $T = B = 0$ for the Minkowski case ($\mathcal{A} = 1$ and $\mathcal{B} = 1$).

Spherical symmetry in $f(T, B)$ gravity

For this tetrad, the $f(T, B)$ field equations yield

$$\frac{1}{2}\kappa^2\rho = \frac{1}{4}f + \frac{r\mathcal{B}(\sqrt{\mathcal{B}}-1)\mathcal{A}' + \mathcal{A}(r\mathcal{B}' + 2\mathcal{B}^{3/2} - 2\mathcal{B})}{2r^2\mathcal{A}\mathcal{B}^2}f_T - \frac{r\mathcal{B}' f_B' - 4\mathcal{B}^{3/2}\left(f_B' + f_T'\right) + 4\mathcal{B}f_T'}{4r\mathcal{B}^2}, \\ + \frac{r^2\mathcal{B}\mathcal{A}'^2 + r\mathcal{A}\left[r\mathcal{A}'\mathcal{B}' + 4\mathcal{B}^{3/2}\mathcal{A}' - 2\mathcal{B}(r\mathcal{A}'' + 4\mathcal{A}')\right] + 4\mathcal{A}^2(r\mathcal{B}' + 2\mathcal{B}^{3/2} - 2\mathcal{B})}{8r^2\mathcal{A}^2\mathcal{B}^2}f_B \\ + \frac{f_B''}{2\mathcal{B}},$$

$$\frac{1}{2}\kappa^2 p_r = -\frac{1}{4}f - \frac{r(\sqrt{\mathcal{B}}-2)\mathcal{A}' + 2\mathcal{A}(\sqrt{\mathcal{B}}-1)}{2r^2\mathcal{A}\mathcal{B}}f_T - \frac{r\mathcal{A}' + 4\mathcal{A}}{4r\mathcal{A}\mathcal{B}}f_B' \\ + \frac{-r^2\mathcal{B}\mathcal{A}'^2 + r\mathcal{A}\left[-r\mathcal{A}'\mathcal{B}' - 4\mathcal{B}^{3/2}\mathcal{A}' + 2\mathcal{B}(r\mathcal{A}'' + 4\mathcal{A}')\right] - 4\mathcal{A}^2\left(r\mathcal{B}' + 2\mathcal{B}^{3/2} - 2\mathcal{B}\right)}{8r^2\mathcal{A}^2\mathcal{B}^2}f_B$$

where primes denote differentiation with respect to the radial coordinate, then, $f_T' = f_{TT}T' + f_{TB}B'$ and $f_B' = f_{BB}B' + f_{TB}T'$

Perturbations around Schwarzschild

- Finding new exact solutions is difficult in $f(T,B)$, there have been some attempts but none of them have found physically interesting solutions so far.†
- Let us then assume that the background is described by the Schwarzschild geometry and the perturbed coefficients are first order corrections to this spacetime, namely,

$$\mathcal{A}(r) = 1 - \frac{2M}{r} + \epsilon\, a(r)\,,$$
$$\mathcal{B}(r) = \left(1 - \frac{2M}{r}\right)^{-1} + \epsilon\, b(r)\,.$$

Here $\epsilon \ll 1$ is a small tracking parameter.

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Model studied

- To include different power-law forms of the Lagrangian, let us assume the following combination of power-law terms

$$f(T,B) = T + \frac{1}{2}\epsilon\left(\alpha T^q + \beta B^m + \gamma B^s T^w + \zeta(\xi T + \chi B)^u\right) ,$$

  where $\alpha, \beta, \gamma, \zeta, q, m, s, w$ and u are constants.
- The case $\beta = \gamma = \zeta = 0$ (power-law $f(T)$) was studied before [‡].

[‡] S. Bahamonde, K. Flathmann and C. Pfeifer, Phys. Rev. D **100** (2019) no.8, 084064 .

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Perturbed solutions

- The Cases 1a-1d have the following solutions:

$$\begin{aligned}\mathcal{A}(r) &= \mu^2 - \frac{1}{(\mu^2-1)^2 r^2}\Big[3\beta\mu^7 - \frac{1}{2}(\alpha+13\beta)\mu^6 - 4\beta\mu^5 + \frac{1}{2}(15\alpha+43\beta)\mu^4 \\ &\quad -\frac{2}{3}(32\alpha+35\beta)\mu^3 + \frac{1}{2}(33\alpha+13\beta)\mu^2 + 4\beta\mu - \frac{1}{6}(13\alpha+\beta) - \frac{\beta}{\mu} \\ &\quad -2(\alpha+\beta)\left(1-3\mu^2\right)\log\mu\Big] + \epsilon\,\tilde{a}_\gamma(r)\,, \\ \mathcal{B}(r) &= \mu^{-2} + \frac{\epsilon}{r^2(\mu^2-1)}\Big[\frac{1}{2}(25\alpha+37\beta) - 4(\alpha+2\beta)\mu - \frac{2(16\alpha+13\beta)}{3\mu} \\ &\quad -\frac{2(\alpha+3\beta)}{\mu^2} + \frac{4(\alpha+\beta)}{\mu^3} + \frac{\alpha-11\beta}{6\mu^4} + \frac{2\beta}{\mu^5} + \frac{2(\alpha+\beta)\log(\mu)}{\mu^4}\Big] + \epsilon\,\tilde{b}_\gamma(r)\,,\end{aligned}$$

where $\mu^2 = 1 - 2M/r$.

- $\tilde{a}_\gamma(r)$ and $\tilde{b}_\gamma(r)$ depend on the model. All the solutions are asymptotically flat.
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Outline

Wordline, energy and momentum

- We can write down the worldline $q(\tau)$ of a test particle as

$$2\mathcal{L} = g_{\mu\nu}\dot{q}^\mu\dot{q}^\nu = \mathcal{A}\,\dot{t}^2 - \mathcal{B}\,\dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}^2 = \sigma\,,$$

  where $q^\mu(\tau) = (t(\tau), r(\tau), \theta(\tau), \phi(\tau))$, $\sigma = 1$ ($\sigma = 0$) for massive particles (massless particles).

- By setting $\theta = \pi/2$, we find that the conserved quantities, energy k and momentum h become, respectively

$$k = \frac{\partial \mathcal{L}}{\partial \dot{t}} = \mathcal{A}\dot{t} = \left(1 - \frac{2M}{r} + \epsilon\, a(r)\right)\dot{t}\,,$$

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Potential

- One gets the potential Eq:

$$\dot{r}^2 + 2V(r) = 0\,,$$

where we have defined the potential as

$$V(r) = -\frac{1}{2}\mathcal{B}^{-1}\left(\frac{k^2}{\mathcal{A}} - \frac{h^2}{r^2} - \sigma\right).$$

- By replacing the metric functions and expanding up to first order in ϵ, we get that the potential becomes

$$V(r) = -\frac{1}{2}k^2 + \frac{1}{2}\left(1 - \frac{2M}{r}\right)\left(\frac{h^2}{r^2} + \sigma\right)$$
$$+ \frac{\epsilon}{2}\left[k^2\left(\frac{a(r)}{1-\frac{2M}{r}} + b(r)\left(1-\frac{2M}{r}\right)\right) - b(r)\left(\sigma + \frac{h^2}{r^2}\right)\left(1-\frac{2M}{r}\right)^2\right].$$

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Photon sphere and Shadow of the black hole

- For circular photon orbits ($\sigma = 0$) we must have that the potential and its derivatives vanish, i.e., $V = V' = 0$.
- We need to expand the radial (circular) coordinate $r_c = r_0 + \epsilon\, r_1$, energy $k = k_0 + \epsilon\, k_1$ and angular momentum $h = h_0 + \epsilon\, h_1$.
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- the shadow of the black hole predicted by power-law $f(T,B)$ will be enlarged or reduced for each solution.
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- The same happens for the γB^2 and γBT (Case 1b), i.e., when $\beta, \gamma > 0$ ($\beta, \gamma < 0$), the shadow will be bigger.
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Perihelion shift

- One can rewrite the potential equation as

$$\frac{1}{2}\frac{\dot{r}^2}{\dot{\phi}^2}+\frac{1}{\dot{\phi}^2}V(r)=\frac{1}{2}\left(\frac{dr}{d\phi}\right)^2+\frac{r^4}{h^2}V(r)=0\,.$$

- One gets that the perihelion shift can be defined as

$$\Delta\phi=2\pi\left(\frac{h}{r_c^2\sqrt{V''(r_c)}}-1\right)\,.$$

- For massive objects: we consider the potential V, V', V''. We evaluate the equations $V(r_c)=0$ and $V'(r_c)=0$ with $h=h_0+\epsilon\, h_1$ and $k=k_0+\epsilon\, k_1$. And then we replace this in the above expressions up to first order in ϵ.

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$$\begin{aligned}\Delta\phi &= \Delta\phi_{\rm GR} + \epsilon\,\Delta\phi_\epsilon \\ &\approx 6\pi q + 27\pi q^2 + 135\pi q^3 + \mathcal{O}(q^4) + \\ &\quad \epsilon\,\pi\Big(\frac{12\beta q}{r_c^2} + \frac{8q^2(\alpha+10\beta)}{r_c^2} + \frac{q^3(194\alpha+1139\beta)}{2r_c^2} + \gamma\Delta\phi_\gamma\Big)\,,\end{aligned}$$

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Deflection of light

- Following a similar computation as before, we find that the deflection angle for all the models can be expressed as

$$\vartheta = \vartheta_{\rm GR} + \epsilon\,\vartheta_\epsilon \approx \frac{4M}{r_0} + \frac{M^2}{r_0^2}\left(\frac{15\pi}{4} - 4\right) + \frac{M^3}{r_0^3}\left(\frac{244 - 45\pi}{6}\right) + \epsilon\left[\frac{4M^3}{15r_0^5}(16\alpha + \beta) + \gamma\vartheta_\gamma\right],$$

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Shapiro Delay

- The Shapiro effect represents the time correction for the round trip of a radar signal that passes near a massive object in the presence of gravity.
- The time retardation for all the solutions can be written as,

$$\begin{aligned} t_{\rm S}(r,r_0) &= t_{\rm S,GR}(r,r_0) + \epsilon\, t_{\rm S,\epsilon}(r,r_0) \\ &\approx M\left[1 + 2\log\left(\frac{2r}{r_0}\right) - \frac{r_0}{r}\right] + \epsilon\left[\frac{8\alpha M^3}{3r_0^4} - \frac{\beta M^2}{r r_0^2} + \gamma\, t_{\rm S,\gamma}\right], \end{aligned}$$

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Constraining the models

- For example, for the perihelion shift of Mercury, for GR the computation gives

$$\Delta\phi_{\rm GR,Mercury} \approx 0.1033\,\text{''/cycles} \approx 42,84\,\text{''/cen}\,.$$

- The observed value is $42,98 \pm 0.040\,$''/cen, so $\Delta\phi_\epsilon$ must lie between the error bars:

$$\Delta\phi_{\epsilon,\rm max} \approx 0.18\,\text{''/cen}\,.$$

- For example, for the Case 1a, one gets that the maximum value that the constants could be are

$$\Big|\alpha + 5.65\times 10^7\beta\Big|_{\rm max} \approx 3.65\times 10^{20}\,{\rm km}^2\,.$$

- We did the same for all the Solar System tests to find the maximum value for the constants (see more details about what "data" we used in our paper)

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Summary of results - Max. values

Model	Perihelion shift	Deflection Light	Cassini	Shapiro delay	Grav. redshift
Case 1a	$\lvert\alpha+10^{8}\beta\rvert \lesssim 10^{20}$	$\lvert\alpha+10^{-1}\beta\rvert \lesssim 10^{19}$	$\alpha+10^{-1}\beta \lesssim 10^{23}$	$\lvert\alpha-10^{3}\beta\rvert \lesssim 10^{21}$	$\alpha+10^{9}\beta \lesssim 10^{22}$
Case 1b	$\lvert-\gamma\rvert \lesssim 10^{13}$	$\lvert\gamma\rvert \lesssim 10^{20}$	$\lvert\gamma\rvert \lesssim 10^{23}$	$\lvert-\gamma\rvert \lesssim 10^{18}$	$\lvert\gamma\rvert \lesssim 10^{13}$
Case 1c	$\lvert-\gamma\rvert \lesssim 10^{42}$	$\lvert\gamma\rvert \lesssim 10^{39}$	$\lvert\gamma\rvert \lesssim 10^{43}$	$\lvert\gamma\rvert \lesssim 10^{40}$	$\lvert-\gamma\rvert \lesssim 10^{38}$
Case 1d	$\lvert\gamma\rvert \lesssim 10^{43}$	$\lvert\gamma\rvert \lesssim 10^{39}$	$\lvert\gamma\rvert \lesssim 10^{43}$	$\lvert\gamma\rvert \lesssim 10^{41}$	$\lvert-\gamma\rvert \lesssim 10^{39}$

Table: Constrains for the different solutions with different Solar System tests only considering the order of magnitudes of the maximum values of the parameters. The values have dimensions of km^2, km^4 depending on the solutions, but we have omitted them here in order to safe space. For each case, we have rounded the numbers to only show their order of magnitude. Cases 1b-1d also contain the same α and β contributions from Case 1a, but we have omitted them for simplicity to only show the order of magnitude in γ. These contributions should also appear in Cases 1b-1d in the same way in Case 1a.

- Even though the numbers in the table look large, they are not dimensionless quantity. Then, it may be made arbitrarily large or small by a simple change of units.
- One should clarify that if the order of magnitude is bigger, this would mean that the constant can take much bigger values, and hence, the contribution coming from that constant is much smaller.

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Model	Perihelion shift	Deflection Light	Cassini	Shapiro delay	Grav. redshift
Case 1a	$\lvert\alpha+10^8\beta\rvert\lesssim 10^{20}$	$\lvert\alpha+10^{-1}\beta\rvert\lesssim 10^{19}$	$\alpha+10^{-1}\beta\lesssim 10^{23}$	$\lvert\alpha-10^3\beta\rvert\lesssim 10^{21}$	$\alpha+10^9\beta\lesssim 10^{22}$
Case 1b	$\lvert-\gamma\rvert\lesssim 10^{13}$	$\lvert\gamma\rvert\lesssim 10^{20}$	$\lvert\gamma\rvert\lesssim 10^{23}$	$\lvert-\gamma\rvert\lesssim 10^{18}$	$\lvert\gamma\rvert\lesssim 10^{13}$
Case 1c	$\lvert-\gamma\rvert\lesssim 10^{42}$	$\lvert\gamma\rvert\lesssim 10^{39}$	$\lvert\gamma\rvert\lesssim 10^{43}$	$\lvert\gamma\rvert\lesssim 10^{40}$	$\lvert-\gamma\rvert\lesssim 10^{38}$
Case 1d	$\lvert\gamma\rvert\lesssim 10^{43}$	$\lvert\gamma\rvert\lesssim 10^{39}$	$\lvert\gamma\rvert\lesssim 10^{43}$	$\lvert\gamma\rvert\lesssim 10^{41}$	$\lvert-\gamma\rvert\lesssim 10^{39}$

Table: Constrains for the different solutions with different Solar System tests only considering the order of magnitudes of the maximum values of the parameters. The values have dimensions of km^2, km^4 depending on the solutions, but we have omitted them here in order to safe space. For each case, we have rounded the numbers to only show their order of magnitude. Cases 1b-1d also contain the same α and β contributions from Case 1a, but we have omitted them for simplicity to only show the order of magnitude in γ. These contributions should also appear in Cases 1b-1d in the same way in Case 1a.

- Even though the numbers in the table look large, they are not dimensionless quantity. Then, it may be made arbitrarily large or small by a simple change of units.
- One should clarify that if the order of magnitude is bigger, this would mean that the constant can take much bigger values, and hence, the contribution coming from that constant is much smaller.

Outline

1. Brief Introduction to Teleparallel theories of gravity
2. New perturbed spherically symmetric solutions
   - Theory and spherical symmetry
   - Perturbed spherically symmetric solutions
3. Phenomenology in $f(T, B)$ gravity
   - Potential and Basic ingredients
   - Solar System Tests
4. Conclusions

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- We computed 6 different Solar System tests for all the models: photon sphere, perihelion shift, deflection of light, Shapiro delay, Cassini experiment and Grav redshift.
- In all the Solar System tests, the constraints are obtained by comparing the extra leading order terms produced by the particular phenomena against the analog GR term.

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