Teleparallel theories of gravity and applications to cosmology

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Outline

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Fundamental variables and characteristic tensors

- In the most general metric-affine setting, the fundamental variables are a **metric** $g_{\mu\nu}$ (10 comp.) as well as the coefficients $\hat{\Gamma}^{\rho}{}_{\mu\nu}$ (64 comp.) of an **affine connection.**

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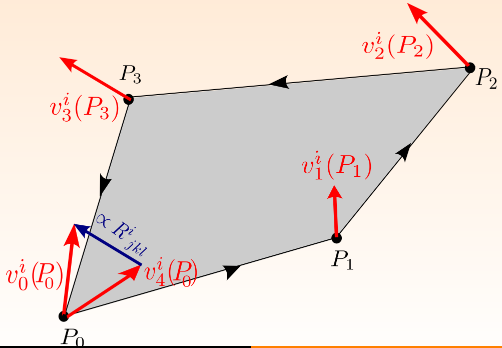
- A priori, there is no relation between $g_{\mu\nu}$ and $\hat{\Gamma}^{\rho}{}_{\mu\nu}$.
- The most general metric-affine theory is characterised by the following tensors:

Curvature	$\hat{R}^{\mu}{}_{\nu\rho\sigma} = \partial_{\rho}\hat{\Gamma}^{\mu}{}_{\nu\sigma} - \partial_{\sigma}\hat{\Gamma}^{\mu}{}_{\nu\rho} + \hat{\Gamma}^{\mu}{}_{\tau\rho}\hat{\Gamma}^{\tau}{}_{\nu\sigma} - \hat{\Gamma}^{\mu}{}_{\tau\sigma}\hat{\Gamma}^{\tau}{}_{\nu\rho}$
Torsion	$\hat{T}^{\mu}{}_{\nu\rho} = \hat{\Gamma}^{\mu}{}_{\rho\nu} - \hat{\Gamma}^{\mu}{}_{\nu\rho}$
Non-metricity	$\hat{Q}_{\mu\nu\rho} = \hat{\nabla}_{\mu}g_{\nu\rho} = \partial_{\mu}g_{\nu\rho} - \hat{\Gamma}^{\sigma}{}_{\nu\mu}g_{\sigma\rho} - \hat{\Gamma}^{\sigma}{}_{\rho\mu}g_{\nu\sigma}$

What does curvature represent geometrically?

Curvature tensor $\hat{R}^{\alpha}{}_{\beta\mu\nu}$

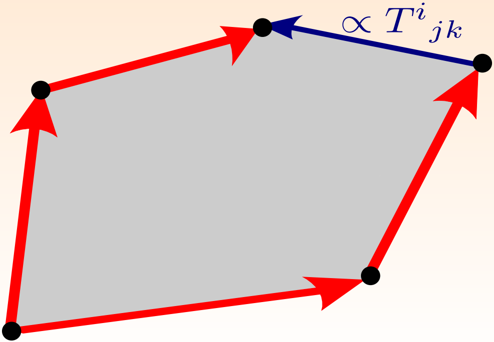
Rotation experienced by a vector when it is parallel transported along a closed curve



What does torsion represent geometrically?

Torsion tensor $\hat{T}^{\alpha}{}_{\mu\nu}$

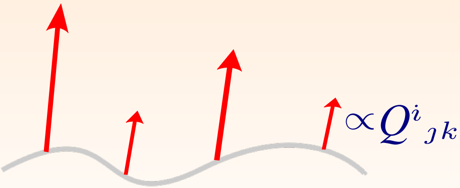
non-closure of the parallelogram formed when two infinitesimal vectors are parallel transported along each other.



What does non-metricity represent geometrically?

Non-metricity tensor $\hat{Q}_{\alpha\mu\nu}$

measures how much the length and angle of vectors change as we parallel transport them, so in metric spaces the length of vectors is conserve



Connection decomposition

A general affine connection can uniquely be decomposed into three parts in the form:

$$\hat{\Gamma}^{\rho}{}_{\mu\nu} = \mathring{\Gamma}^{\rho}{}_{\mu\nu} + \hat{K}^{\rho}{}_{\mu\nu} + \hat{L}^{\rho}{}_{\mu\nu}\,,$$

where the quantities appearing above are

Levi-Civita connection	$\mathring{\Gamma}^{\mu}{}_{\nu\rho} = \frac{1}{2}g^{\mu\sigma}\left(\partial_{\nu}g_{\sigma\rho} + \partial_{\rho}g_{\nu\sigma} - \partial_{\sigma}g_{\nu\rho}\right)$
Contortion tensor	$\hat{K}^{\mu}{}_{\nu\rho} = \frac{1}{2}\left(\hat{T}_{\nu}{}^{\mu}{}_{\rho} + \hat{T}_{\rho}{}^{\mu}{}_{\nu} - \hat{T}^{\mu}{}_{\nu\rho}\right)$
Disformation tensor	$\hat{L}^{\mu}{}_{\nu\rho} = \frac{1}{2}\left(\hat{Q}^{\mu}{}_{\nu\rho} - \hat{Q}_{\nu}{}^{\mu}{}_{\rho} - \hat{Q}_{\rho}{}^{\mu}{}_{\nu}\right)$

Some important special cases of metric-affine geometries

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- **General Teleparallel geometry** ($\hat{R}_{\alpha\mu\nu\beta}=0$): In the case of vanishing curvature, the connection is flat.

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- **Symmetric Teleparallel geometry** ($\hat{R}_{\alpha\mu\nu\beta}=0, \hat{T}^{\alpha}_{\mu\nu}=0$): Both torsion tensor and curvature are zero and the gravitational interactions are only mediated through non-metricity.

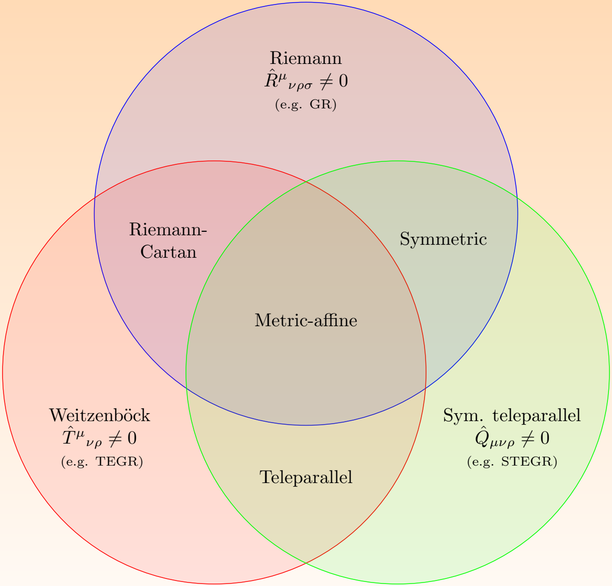

Figure: Classification of metric-affine geometries

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- In Teleparallel gravity, the spin connection $\omega^a{}_{b\mu}$ is associated solely with **inertial effects**.

Connection in Teleparallel gravity

- GR assumes zero torsion and non-zero curvature (Levi-Civita connection) whereas Teleparallel gravity (TG) assumes **zero curvature and non-zero torsion.** Both assumes non-metricity to be zero, meaning $\hat{\nabla}_{\alpha} g_{\mu\nu} = 0$.

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- The pure tetrad formalism was the initial framework used for TG, which chooses a specific frame where the spin connection $\omega^a{}_{b\mu}$ vanishes. Be careful choosing the correct tetrad which is compatible with this gauge.

Local Lorentz transformations

- If one performs a local Lorentz transformation ${e'}^{a}{}_{\mu} = \Lambda^{a}{}_{b} e^{b}{}_{\mu}$, the metric $g_{\mu\nu} = g'_{\mu\nu}$ is invariant. A consequence of this is that the metric has 10 d.o.f. and the tetrads 10+6(the extra comes from $\Lambda^{a}{}_{b}$). Different tetrads can give the same metric.

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- The torsion tensor is covariant under local Lorentz transformation.

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Curvature in Teleparallel gravity

$$R^{a}{}_{b\mu\nu}(\omega^{a}{}_{b\mu}) = \partial_{\mu}\omega^{a}{}_{b\nu} - \partial_{\nu}\omega^{a}{}_{b\mu} + \omega^{a}{}_{c\mu}\omega^{c}{}_{b\nu} - \omega^{a}{}_{c\nu}\omega^{c}{}_{b\mu} \equiv 0\,.$$

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- **Be careful here!** The general curvature $R^\lambda{}_{\mu\sigma\nu} \equiv 0$, not $\mathring{R}^\lambda{}_{\mu\sigma\nu} \neq 0$

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- The Ricci scalar computed from the Levi-Civita connection $\mathring{R}$ differs from the scalar torsion T by a boundary term B.

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Equivalence between field equations

The field equations arising from $S_{\rm TEGR}$ are equivalent to the Einstein field equations.

Two different ways of understanding gravity

Coupling to matter

In TG, no direct matter coupling to the teleparallel connection is introduced, in order to preserve the weak equivalence principle $\Longrightarrow$ matter fields retain their universal **coupling to the metric** and possibly its **Levi-Civita connection** (in the case of spinor fields).

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Equivalence on their field equations

TEGR has the same equations as GR, so **CLASSICALLY** it is impossible to make any observation to distinguish between them. All **classical experiments** already done, that have confirmed GR, also can be understood as a confirmation of TEGR.

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Ricci scalar and non-metricity

$$\overset{\diamond}{R} = \overset{\circ}{R} + \overset{\diamond}{Q} + \mathcal{D}_\alpha(\overset{\diamond}{Q}{}^\alpha - \tilde{\overset{\diamond}{Q}}{}^\alpha) = 0 \rightarrow \overset{\circ}{R} = -\overset{\diamond}{Q} + \mathcal{D}_\alpha(\tilde{\overset{\diamond}{Q}}{}^\alpha - \overset{\diamond}{Q}{}^\alpha)\,,$$

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where $\overset{\diamond}{Q} = \frac{1}{4}\overset{\diamond}{Q}_{\alpha\beta\gamma}Q^{\alpha\beta\gamma} - \frac{1}{2}\overset{\diamond}{Q}_{\alpha\beta\gamma}Q^{\beta\alpha\gamma} - \frac{1}{4}\overset{\diamond}{Q}_\alpha\overset{\diamond}{Q}^\alpha + \frac{1}{2}\overset{\diamond}{Q}_\alpha\tilde{\overset{\diamond}{Q}}^\alpha$, $\overset{\diamond}{Q}_\alpha = \overset{\diamond}{Q}_{\alpha\lambda}{}^\lambda$ and $\tilde{\overset{\diamond}{Q}}_\alpha = \overset{\diamond}{Q}^\lambda{}_{\lambda\alpha}$.

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- Since $\overset{\diamond}{Q}$ and $\overset{\circ}{R}$ differs by a boundary term, the above action also gives the Einstein field equations.

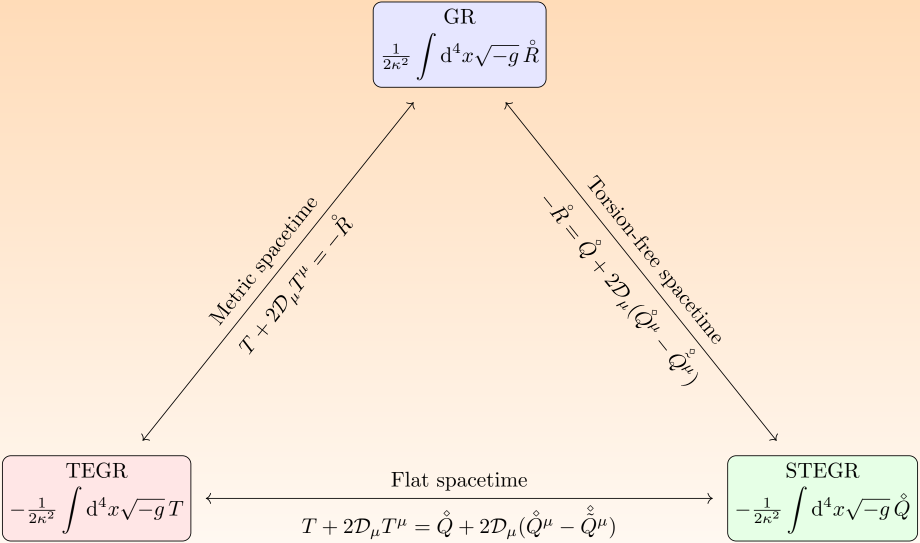

Figure: Geometrical trinity of gravity

Outline

Modified teleparallel theories

What happens if we modify TEGR?

If we modify the TEGR action, a priori there is no equivalence between modified theories from GR and modified Teleparallel theories.

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Are Teleparallel theories Lorentz invariant?

There is a lot of misconceptions in the literature since TEGR and their first modifications were proposed in the pure-tetrad formalism, which assumes that ${\omega^{a}}_{b\mu} = 0$ globally. By assuming this choice, TEGR is pseudo local Lorentz invariant (invariant up to a boundary term) and in modified TG, there is a breaking of the local Lorentz invariant.

Modified teleparallel theories

Is it possible to formulate Teleparallel theories in a fully invariant form?

Yes, and the way to do this is by incorporating the spin connection $\omega^{a}{}_{b\mu}$ in the formulation. If one introduces this quantity, the torsion tensor is always fully covariant (covariant under both diffeo and local Lorentz) and then, any action constructed from it will be fully invariant.

See[1]

[1] M. Krššák and E. N. Saridakis, Class. Quant. Grav. **33** (2016) no.11, 115009; M. Krssak, R. J. van den Hoogen, J. G. Pereira, C. G. Böhmer and A. A. Coley, Class. Quant. Grav. **36** (2019) no.18, 183001.

Important properties of Teleparallel theories

- **Gauge nature:** Teleparallel gravity can be written as the gauge theory of translations with the torsion tensor being the field strength of the theory.

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- **The division between inertial and gravitational effects:** In GR, gravity and inertia cannot be separated, but in TG, there are some arguments in favour that it is possible to separate them by putting all the inertial effects in ${\omega^a}_{b\mu}$ and gravity in the tetrads ${e^a}_\mu$.

Important properties of Teleparallel theories

- Teleparallel theories have the tetrads and spin connection as the fundamental variables, so that, one most commonly assumes an action which is of the form

$$\mathcal{S} = \mathcal{S}_{\mathsf{g}}[e,\omega] + \mathcal{S}_{\mathsf{m}}[e,\chi]\,,$$

where the gravitational part $\mathcal{S}_{\mathsf{g}}$ of the action depends on the tetrad $e^{A}{}_{\mu}$ and the spin connection $\omega^{A}{}_{B\mu}$, while the matter part depends on the tetrad $e^{A}{}_{\mu}$ and arbitrary matter fields χ^{I}, but not on the spin connection.

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- Particles (bosonic or femionic) follow the standard geodesic equation.

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- Variations of any action should be taken with respect to both variables (with the emphasis that the spin connection is non-arbitrary but always flat).
- Since $\omega^{A}{}_{B\mu}$ is a pure-gauge quantity, it can be shown that the the antisymmetric part of the field equations arising from variations w/r to the tetrads $e^{A}{}_{\mu}$ coincides with the variations of the action w/r to $\omega^{A}{}_{B\mu}$.

Important properties of Teleparallel theories

- Field equations are fully covariant (Lorentz and diffeo), and after finding them, it is possible to choose a gauge, known as the Weitzenböck gauge which is a special frame where $\omega^{A}{}_{B\mu} = 0$.

Important properties of Teleparallel theories

- Field equations are fully covariant (Lorentz and diffeo), and after finding them, it is possible to choose a gauge, known as the Weitzenböck gauge which is a special frame where $\omega^{A}{}_{B\mu} = 0$.
- This gauge choice can be only taken after deriving the field equations and if one does this, only some tetrads will be compatible with this choice.

New General Relativity (NGR)

The torsion tensor can be decomposed in its irreducible parts as

$$a_\mu = \frac{1}{6}\epsilon_{\mu\nu\sigma\rho}T^{\nu\sigma\rho}\,, \quad v_\mu = T^\sigma{}_{\sigma\mu}\,,$$
$$t_{\sigma\mu\nu} = \frac{1}{2}\left(T_{\sigma\mu\nu} + T_{\mu\sigma\nu}\right) + \frac{1}{6}\left(g_{\nu\sigma}v_\mu + g_{\nu\mu}v_\sigma\right) - \frac{1}{3}g_{\sigma\mu}v_\nu\,,$$

where $\epsilon_{\mu\nu\sigma\rho}$ is the totally anti-symmetric Levi-Civita symbol. From these we build the scalars

$$T_{\text{ax}} = a_\mu a^\mu\,, \quad T_{\text{vec}} = v_\mu v^\mu\,, \quad T_{\text{ten}} = t_{\sigma\mu\nu}t^{\sigma\mu\nu}\,,$$

and the torsion scalar is a linear combination

$$T = \frac{3}{2}T_{\text{ax}} + \frac{2}{3}T_{\text{ten}} - \frac{2}{3}T_{\text{vec}}\,.$$

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- Perturbations around Minkowski shows that the unique stable Minkowski background that includes gravity is the TEGR case[3].

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- **Strongly coupling problem?** By performing Minkowski perturbations, one only finds new modes at 4th order in the perturbation (J. Beltrán Jiménez, A. Golovnev, T. Koivisto and H. Veermäe, [arXiv:2004.07536])

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- Other theories related to the boundary term such as $-T + f(B)$ gravity.

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Condition 3

The field equations must be covariant under local Lorentz transformations.

Conditions for the theory

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Any number of contractions of the irreducible parts of the torsion tensor will result in second order field equations. This means that an infinite number of terms can be formed in Teleparallel gravity that give rise to second order field equations. However, it is unclear how physical such higher order contributions will be.

Teleparallel Horndeski gravity

- Taking quadratic contractions of the torsion tensor, the most general Lagrangian of Teleparallel gravity satisfying the conditions turns out to be $f(T_{\text{ax}}, T_{\text{vec}}, T_{\text{ten}})$[6]

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Possible independent scalars

$$I_2 = v^{\mu}\phi_{;\mu}\,, \quad J_1 = a^{\mu}a^{\nu}\phi_{;\mu}\phi_{;\nu}\,, \quad J_3 = v_{\sigma}t^{\sigma\mu\nu}\phi_{;\mu}\phi_{;\nu}\,,$$
$$J_5 = t^{\sigma\mu\nu}t_{\sigma}{}^{\bar{\mu}}{}_{\nu}\phi_{;\mu}\phi_{;\bar{\mu}}\,, \quad J_6 = t^{\sigma\mu\nu}t_{\sigma}{}^{\bar{\mu}\bar{\nu}}\phi_{;\mu}\phi_{;\nu}\phi_{;\bar{\mu}}\phi_{;\bar{\nu}}\,,$$
$$J_8 = t^{\sigma\mu\nu}t_{\sigma\mu}{}^{\bar{\nu}}\phi_{;\nu}\phi_{;\bar{\nu}}\,, \quad J_{10} = \epsilon^{\mu}{}_{\nu\rho\sigma}a^{\nu}t^{\alpha\rho\sigma}\phi_{;\mu}\phi_{;\alpha}\,.$$

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Teleparallel Horndeski action

$$\mathcal{S}_{\rm Tele-deski} = \int d^4x \Big[\mathcal{L}_{\text{Horndeski}} + \mathcal{L}_{\text{Tele}}\Big] e = \int d^4x \Big[\sum_{i=2}^{5} \mathcal{L}_i + G_{\text{Tele}}\Big] e \,,$$

where[7]

$$\begin{aligned}
G_{\text{Tele}} &= G_{\text{Tele}}(\phi, X, T, T_{\text{ax}}, T_{\text{vec}}, I_2, J_1, J_3, J_5, J_6, J_8, J_{10})\,,\\
\mathcal{L}_2 &= G_2(\phi, X)\,, \quad \mathcal{L}_3 = G_3(\phi, X)\mathring{\Box}\phi\,,\\
\mathcal{L}_4 &= G_4(\phi, X)\left(-T + B\right) + G_{4,X}(\phi, X)\left[\left(\mathring{\Box}\phi\right)^2 - \phi_{;\mu\nu}\phi^{;\mu\nu}\right]\,,\\
\mathcal{L}_5 &= G_5(\phi, X)\mathcal{G}_{\mu\nu}\phi^{;\mu\nu} - \frac{1}{6}G_{5,X}(\phi, X)\Big[\left(\mathring{\Box}\phi\right)^3 + 2\phi_{;\mu}{}^{\nu}\phi_{;\nu}{}^{\alpha}\phi_{;\alpha}{}^{\mu}\\
&\quad - 3\phi_{;\mu\nu}\phi^{\mu\nu}\left(\mathring{\Box}\phi\right)\Big]\,.
\end{aligned}$$

[7] S. Bahamonde, K. F. Dialektopoulos and J. Levi Said, Phys. Rev. D **100** (2019) no.6, 064018.

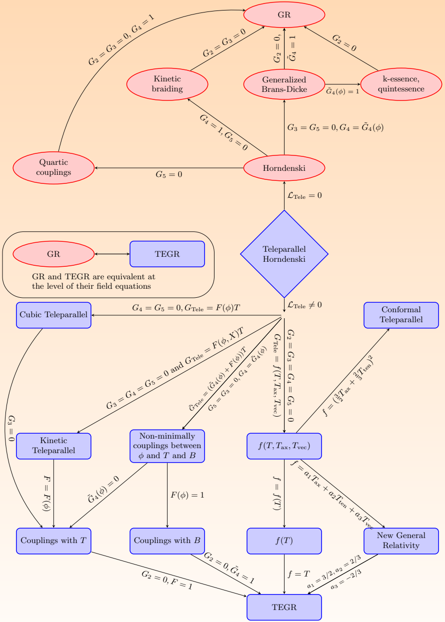

FIG. 1: Relationship between Teleparallel Horndenski and various theories.

Outline

How to work with different geometric symmetries

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- Teleparallel theories have the tetrads and spin connection (being always flat) as the fundamental variables
- Then, for a specific symmetry, we assume that both the spin connection(flat) and the tetrad satisfy the symmetry condition $L_\xi \omega^a{}_{b\mu} = 0$ and $L_\xi e^a{}_\mu = 0$.
- For a flat FLRW spacetime in Cartesian coordinates (t, x, y, z), this condition is satisfied for $\omega^a{}_{b\mu} = 0$ and

FLRW tetrad compatible with cosmological symmetries in the Weitzenböck gauge

$$e^a{}_\mu = \mathrm{diag}(N(t), a(t), a(t), a(t))$$
$$\to ds^2 = N(t)^2 - a(t)^2(dx^2 + dy^2 + dz^2)\,.$$

How to work with different geometric symmetries

Antisymmetric field equations

Important point: the tetrad showed in the last slide in the Weitzenböck gauge solves all the antisymmetric field equations for any Teleparallel gravitational theory

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Spherical coordinates

Be careful here: In spherical coordinates (t, r, θ, ϕ), the tetrad in the Weitzenböck gauge looks more complicated (off-diagonal terms appear):

$$e^a{}_\mu = \begin{pmatrix} N(t) & 0 & 0 & 0 \\ 0 & a(t)\sin(\theta)\cos(\phi) & ra(t)\cos(\theta)\cos(\phi) & -ra(t)\sin(\theta)\sin(\phi) \\ 0 & a(t)\sin(\theta)\sin(\phi) & ra(t)\cos(\theta)\sin(\phi) & ra(t)\sin(\theta)\cos(\phi) \\ 0 & a(t)\cos(\theta) & -ra(t)\sin(\theta) & 0 \end{pmatrix}.$$

Cosmological perturbations in TG

- In a metrical theory, one perturbs the FLRW metric in the scalar-vector-tensor decomposition form:

$$\delta g_{\mu\nu} = \begin{bmatrix} -2\varphi & a\left(\partial_i \mathcal{B} + \mathcal{B}_i\right) \\ a\left(\partial_i \mathcal{B} + \mathcal{B}_i\right) & 2a^2\left(-\psi\delta_{ij} + \partial_i\partial_j h + 2\partial_{(i}h_{j)} + \frac{1}{2}h_{ij}\right) \end{bmatrix}.$$

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- In TG, the zeroth-order is $e^a{}_\mu = \mathrm{diag}(N(t), a(t), a(t), a(t))$ with $\omega^a{}_{b\mu} = 0$. We perturb this tetrad reproducing the above metric:

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- The **metric has 10 d.o.f.** (4 scalars(1 each), 2 vectors(2 each), 1 tensor(2 each)) and the **tetrads 16 d.o.f.** (6 scalars(1 each), 4 vectors(2 each), 1 tensor(2 each)).

Background cosmology in $f(T,B)$ gravity

- The modified FLRW equations in $f(T,B)$ gravity are[8]

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\begin{aligned}
-3H^2(3f_B+2f_T)+3H\dot{f}_B-3\dot{H}f_B+\frac{1}{2}f &= \kappa^2\rho_m\,,\\
-\left(3H^2+\dot{H}\right)(3f_B+2f_T)-2H\dot{f}_T+\ddot{f}_B+\frac{1}{2}f &= -\kappa^2 p_m\,.
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Cosmological perturbations in $f(T,B)$ gravity

- **Tensorial perturbations:** GW propagation equation is[9]

$$\ddot{h}_{ij} + (3+\alpha_M)H\dot{h}_{ij} + \frac{k^2}{a^2}h_{ij} = 0\,,$$

meaning that $c_T^2 = 1$ with a Planck mass run rate $\alpha_M = \frac{1}{H}\frac{\dot{f}_T}{f_T}$. Thus, $f_T < 0$ is required for stability issues.

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- **Scalar perturbations:** Very messy symmetric and antisymmetric field equations. The density parameter and the weak lensing parameter in Fourier space of the sub-horizon limit obey

$$\ddot{\delta}_m + 2H\dot{\delta}_m \simeq -\frac{k^2\varphi}{a^2} = 4\pi\rho G_{\rm eff}\delta_{\rm m} = \frac{\kappa^2}{2}\rho G_{\rm eff}\delta_{\rm m}\,,$$

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- There are different branches having different $G_{\rm eff}$ depending on the form of f.
- For example for $f_{BB}+2f_{TB}+f_{TT}=0$ one finds $G_{\rm eff}=-G\frac{4}{3(f_T+12H^2f_{TB})}$. One can use these results to constrain models.

Cosmological perturbations in $f(T,B)$ gravity - H_0 tension

- $f(T)$ gravity model does not show tension on the H_0 that prevails in the ΛCDM cosmology, however, σ_8 tension persists(R. C. Nunes, JCAP **05** (2018), 052)

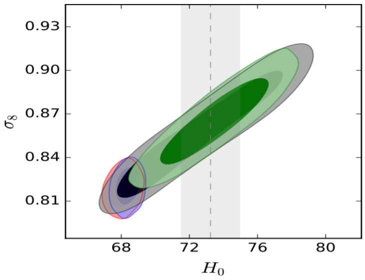

Figure 4. Parametric space in the plane H_0 - σ_8, where the regions in red (blue) show the constraints for ΛCDM model from CMB + BAO (CMB + BAO + H_0), respectively. The regions in black (green) show the constraints for $f(T)$ gravity from CMB + BAO (CMB + BAO + H_0), respectively. The vertical gray band corresponds to $H_0 = 73.24 \pm 1.74$ km s^{-1} Mpc^{-1}.

Background cosmology in Teleparallel scalar-tensor

- Teleparallel dark energy[10] (coupling like $\xi\phi^2 T$) exhibits a quintessence-like, dark-energy-dominated solution, or to the stiff dark-energy late-time attractor, similarly to standard quintessence. There is an additional late-time solution, in which dark energy behaves like a cosmological constant.

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- Theories with a coupling $\chi\phi^2 B$ have late time accelerating attractor solution without requiring any fine tuning of the parameters. A dynamical crossing of the phantom barrier is also possible[11]
- TG non-local cosmology with a term like $Tf(\mathring{\Box}^{-1}T)$ in the action is consistent with the present cosmological data obtained by SNe Ia + BAO + CC + H0 observations[12]

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Teleparallel Horndeski gravity - perturbations

- By considering tensorial perturbations only and after some cumbersome calculations, one gets the following wave equation

$$\ddot{h}_{ij} + (3+\alpha_{\mathsf{M}})\, H\dot{h}_{ij} - (1+\alpha_{\mathsf{T}})\, \frac{k^2}{a^2} h_{ij} = 0,$$

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where $\alpha_T = c_T^2 - 1$ and the speed of GW being equal to[13]

Speed of GW in Teleparallel Horndeski

$$c_T^2 = \frac{G_4 - X(\ddot{\phi}G_{5,X} + G_{5,\phi}) - G_{\mathrm{Tele,T}}}{G_4 - 2XG_{4,X} - X(H\dot{\phi}G_{5,X} - G_{5,\phi}) + 2XG_{\mathrm{Tele,J_8}} + \frac{1}{2}XG_{\mathrm{Tele,J_5}} - G_{\mathrm{Tele,T}}}\,.$$

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Reviving Horndeski using Teleparallel gravity

- For $G_{\text{Tele}} = 0$ (standard case), one gets that to achieve a theory consistent with the GW observations $c_T = 1$, one requires $G_5(\phi, X) = \text{constant}$ and $G_4(\phi, X) = G_4(\phi)$. Hence, Horndeski gravity is highly constraint.

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Teleparallel Lagrangian respecting $c_T=1$ $(\alpha_T=0)$

$$\mathcal{L}=\tilde{G}_{\rm tele}(\phi,X,T,T_{\rm vec},I_2)+\sum_{i=2}^{4}\mathcal{L}_i+G_5(\phi)\mathcal{G}_{\mu\nu}\phi^{;\mu\nu}\,.$$

Outline

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- It is possible to formulate a theory which is equivalent to GR, and then, one can modify these equations to explain dark energy or inflation.
- One needs to be more careful than in Riemannian theories since the tetrad and spin connection form a pair that always need to be considered in a proper way to fulfill the symmetry condition to then solve the antisymmetric field equations.

Conclusions

- Two important TG theories: $f(T,B)$ (contains $f(\mathring{R})$) and Teleparallel Horndeski (contains many scalar-tensor theories).

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- TG cosmology can explain dark energy, alleviate H_0 tension and there are may interesting models with interesting features.
- There are many things totally unexplored in TG, so please go ahead!

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- Cosmology in Extended Gravity (dynamical system analysis, observations and constraints, dark energy, dark matter, inflation, early universe, galaxies, ...)
- Beyond Lorentzian Geometry in Classical and Quantum Gravity (doubly/deformed relativity, standard model extension, Hamilton geometry, Finsler geometry, ...).