Different cosmological behaviours for different frames of $F(R)$ gravity

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28th February 2017

Outline

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1. Introduction
   - Jordan and Einstein frames
   - FRW cosmology in Jordan and Einstein frames
2. Correspondence of $F(R)$ Gravity Singularities in different frames
   - $F(R)$ gravity and scalar-tensor theory
   - Unimodular $F(R)$ gravity and scalar-tensor theory
3. Acceleration and Deceleration in different frames
   - Minimally curvature-coupled scalar-tensor theory
   - Non-minimally curvature-coupled scalar theory
4. Conclusions

$F(R)$ gravity (Jordan frame)

- A well studied modification of GR is $F(R)$ gravity, which has the following action

$F(R)$ gravity action

$$S_{F(R)} = \frac{1}{2\kappa^2} \int F(R)\sqrt{-g}\, d^4x \,.$$

- Here, F is an arbitrary (sufficiently smooth) function of the Ricci scalar. G.R. is recovered if $F(R) = R$.
- Ricci scalar depends on second derivatives of the metric tensor $\rightarrow$ **Fourth order theory.**

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$F(R)$ gravity and a minimally coupled scalar tensor theory

- We will now conformally transform the later action in order to obtain the scalar-tensor Einstein frame counterpart theory.
- By introducing two auxiliary fields A and B, one can re-write the $F(R)$ action as

$$S = \frac{1}{2\kappa^2}\int d^4x\,\sqrt{-g}\left\{B(R-A)+F(A)\right\}\,. \tag{1}$$

- By varying with respect to $B \Longrightarrow A = R$.
- By varying with respect to $A \Longrightarrow B = F'(A)$.

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- Hence, the later action takes the following form

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left\{ F'(A)(R-A) + F(A) \right\} \ . \qquad (2)$$

- Now, we will conformally transform the metric as

Conformal Transformation

$$\tilde{g}_{\mu\nu} = \frac{1}{F'(A)} g_{\mu\nu} = e^{\sigma} g_{\mu\nu} \ , \qquad (3)$$

which modifies the Ricci scalar $R \to \tilde{R}$. Here, $\sigma = -\ln F'(A)$ is a new scalar field defined in terms of A. Tildes refer to the Einstein frame.

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- By using this transformation, the action takes the following form,

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-\tilde{g}} \left\{ \tilde{R} - \frac{3}{2} \tilde{g}^{\mu\nu} \tilde{\partial}_\mu \sigma \tilde{\partial}_\nu \sigma - V(\sigma) \right\} , \quad (4)$$

  where the energy potential was defined as
  $V(\sigma) = \frac{A}{F'(A)} - \frac{F(A)}{F'(A)^2}$.

- **Notation reminder: Variables with tildes mean Einstein frame scalar-tensor theory and without tildes Jordan $F(R)$ frame.**

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- We can rescale the scalar field σ to put it into the canonical form by introducing $\phi = \sqrt{\frac{3}{2\kappa^2}}\sigma$, giving us the following scalar-tensor canonical scalar field action

Minimally coupled scalar-tensor theory (Einstein frame)

$$S = \int d^4x \sqrt{-\tilde{g}} \left\{ \frac{\tilde{R}}{2\kappa^2} - \frac{1}{2}\tilde{\partial}_\mu \phi \tilde{\partial}^\mu \phi - V(\phi) \right\} . \tag{5}$$

- Hence, the above action represents the resulting Einstein frame scalar tensor theory corresponding to the Jordan frame $F(R)$ gravity.
- Conversely, one can start with the above action (Einstein frame) and transform it to a Jordan frame $F(R)$ gravity via $\tilde{g}_{\mu\nu} = e^{-\sigma} g_{\mu\nu} = e^{\mp\sqrt{\frac{2}{3}}\kappa\phi} g_{\mu\nu}$

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- Since both theories are equivalent, depending on the problem studied, it might be convenient to do calculations in one frame than in the other.
- For example, in inflation is easier to work out within the context of scalar-tensor theory.
- We can ask the following question

Important question

These theories are equivalent but, are they actually physically equal?

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Flat Friedmann-Robertson-Walker Cosmology

In this talk, I will focus on a flat FRW background given by the following line-element

Flat FRW metric

$$ds_J^2 = -dt^2 + a(t)^2 \sum_{i=1,2,3} \left(dx^i\right)^2 , \quad (6)$$

where $a(t)$ is the scale factor of the universe in the Jordan $F(R)$ frame.
The homogeneous and isotropic principle implies the above space-time.

FRW Transformations between frames

- By performing the conformal transformation $g_{\mu\nu} \to e^{\pm\sqrt{\frac{2}{3}}\kappa\phi}\tilde{g}_{\mu\nu}$, the FRW metric will be

Line-elements after a conformal transformation

$$ds_J^2 = e^{\pm\sqrt{\frac{2}{3}}\kappa\phi}\left(-d\tilde{t}^2 + \tilde{a}(\tilde{t})^2 \sum_{i=1,2,3} \left(dx^i\right)^2\right), \tag{7}$$

FRW Transformations between Jordan and Einstein frames

$$dt = e^{\pm\frac{1}{2}\sqrt{\frac{2}{3}}\kappa\phi} d\tilde{t}\,, \tag{8}$$

$$a(t(\tilde{t})) = e^{\pm\frac{1}{2}\sqrt{\frac{2}{3}}\kappa\phi}\tilde{a}(\tilde{t})\,. \tag{9}$$

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FRW equations in both frames

FRW equations for scalar-tensor Einstein frame

$$3\tilde{H}^2 = \frac{1}{2}\dot{\phi}^2 + V\,, \tag{10}$$

$$3\tilde{H}^2 + 2\dot{\tilde{H}} = -\frac{1}{2}\dot{\phi}^2 + V\,. \tag{11}$$

FRW equations for $F(R)$ Jordan frame

$$0 = -\frac{F(R)}{2} + 3\left(H^2 + H'\right)F_R(R) - 18\left(4H^2H' + HH''\right)F_{RR}(R)\,, \tag{12}$$

$$\begin{aligned} 0 = {} & \frac{F(R)}{2} - \left(H' + 3H^2\right)F_R(R) + 6\left(8H^2H' + 4H'^2 + 6HH'' + H'''\right)F_{RR}(R) \\ & +36\left(4HH' + H''\right)^2 F_{RRR}(R)\,. \end{aligned} \tag{13}$$

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Do singularities change from one frame to another?

- Having in mind that one can transform from one frame to another by a conformal transformation, we will study what happens with cosmological time singularities when we conformally transform from one frame to another.
- The following question arise:

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- Note: For simplicity, I will set $\kappa = 1$.

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Classification of types of singularities

There are four types of finite time singularities which depends on $a(t), p_{\text{eff}}, \rho_{\text{eff}}$ and derivatives of the Hubble rate:

- Type I ("the Big Rip Singularity"): $a(t)$, ρ_{eff} and p_{eff} diverge as $t \to t_s$. (most severe)
- Type II ("Sudden Singularity"): $a(t)$ and ρ_{eff} remain bounded and p_{eff} diverges as $t \to t_s$.
- Type III: $a(t)$ remains bounded and ρ_{eff} and p_{eff} diverge as $t \to t_s$.
- Type IV: $a(t)$, ρ_{eff} and p_{eff} remain bounded but the second or higher derivatives of the Hubble rate diverge as $t \to t_s$ (least severe).

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Example I: Power-law cosmology from Einstein frame

- Let us consider a power-law cosmology $\tilde{a}(\tilde{t}) = \tilde{a}_0(\tilde{t}/\tilde{t}_0)^p$ where $\tilde{t}_0$ being some fiducial time and p a positive real free parameter. By using the FRW scalar-tensor eqs (Einstein frame) we can easily find that

$$\phi = \pm\sqrt{2p}\ln(\tilde{t}/\tilde{t}_0) \tag{14}$$

- In this case $\tilde{t}_1 = 0$ and $\tilde{t}_2 = \infty$ and in this model the Hubble rate $\tilde{H}$ diverges at $\tilde{t} = 0$, so we have a Type III singularity in the Einstein frame.

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- Let us now apply a conformal transformation to convert the theory to a Jordan frame. If we use the transformations found it before, we get

$$\frac{dt}{d\tilde{t}} = (\tilde{t}/\tilde{t}_0)^{\pm\sqrt{\frac{p}{3}}}, \rightarrow \quad t = \frac{3}{3 \pm \sqrt{3p}} \tilde{t} \left(\frac{\tilde{t}}{\tilde{t}_0}\right)^{\pm 2\sqrt{\frac{p}{3}}}, \qquad (15)$$

$$\Longrightarrow a(t) \sim t^{\frac{\sqrt{3p} \pm 3p}{\sqrt{3p} \pm 3}}. \qquad (16)$$

- For $-$: Type I finite time singularity at $t = 0$ if the power law parameter p lies in the range $1/3 \leq p < 3$.
- If $p = 1/3$, the Jordan frame metric becomes static and there is no longer a singularity.
- In all other cases, the Type III singularity at $\tilde{t} = 0$ in the original Einstein frame remains a type III singularity at $t = 0$ in the Jordan frame.

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- The simplest singular cosmology is described by the following Hubble rate,

$$\tilde{H}(\tilde{t}) = f_0(\tilde{t} - \tilde{t}_s)^\alpha \, , \tag{17}$$

with f_0 an arbitrary real and positive parameter and α a real number

- When $\alpha < -1$, the cosmology develops a Type I singularity.
- When $-1 < \alpha < 0$, the cosmology develops a Type III singularity.
- When $0 < \alpha < 1$, the cosmology develops a Type II singularity.
- When $\alpha > 1$, the cosmology develops a Type IV

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with f_0 an arbitrary real and positive parameter and α a real number

- When $\alpha < -1$, the cosmology develops a Type I singularity.
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- In the Einstein frame we find $\phi = \frac{2\sqrt{-2f_0\alpha}(\tilde{t}-\tilde{t}_s)^{\frac{\alpha+1}{2}}}{\alpha+1}$, so $f_0\alpha < 0$ is required.
- In this example, the solution to $dt = e^{\frac{1}{2}\sqrt{\frac{2}{3}}\phi}d\tilde{t}$ is an incomplete gamma function, so we will express all the quantities depending on $\tilde{t}$ (even the ones in the Jordan frame)
- The transformation blows up at $\tilde{t}_s$ if $\alpha < -1$, so extra caution is required in this case (ϕ will diverge).
- The new Hubble rate (Jordan frame) in terms of the time coordinate $\tilde{t}$ (Einstein frame) will be

$$H(\tilde{t}) = \frac{\sqrt{-f_0\alpha}(\tilde{t}-\tilde{t}_s)^{\frac{\alpha-1}{2}}}{\sqrt{3}} + f_0(\tilde{t}-\tilde{t}_s)^{\alpha} \,. \quad (18)$$

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- Suppose that, $t = f(\tilde{t})$. When, $f(\tilde{t}) \neq 0$, we have,

$$f'(\tilde{t}) = e^{\frac{1}{2}\sqrt{\frac{2}{3}}\phi}, \qquad (19)$$

  and hence $\frac{dH}{dt} = e^{-\frac{1}{2}\sqrt{\frac{2}{3}}\phi}\frac{d\tilde{H}}{d\tilde{t}}$.
- This means that the expression $\frac{dH}{dt}$ diverges if and only if $\frac{d\tilde{H}}{d\tilde{t}}$ diverges for $\alpha > -1$, as then the conformal factor is finite at $\tilde{t}_s$.

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- We know that a Type I singularity occurs when $H(t)$ diverges, so does this singularity still appear at a finite time?
- To respond this, we need to investigate whether $f(\tilde{t}_s)$ is finite. It is easy to show that,

$$t_s = f(\tilde{t}_s) = c_1 - c_2 \Gamma\left(\frac{2}{\alpha+1}\right) , \tag{20}$$

which is finite, provided $\frac{2}{1+\alpha}$ is not a negative integer.

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Hence, we can say:

- If $n \geq 2$ is an integer (with $\alpha \neq 2/n - 1$), the singularity appears in the Hubble rate H at a finite time.
- Reminder: Type II singularity occurs if $\frac{dH}{dt}$ diverges, but H does not diverge.
- Combined together, these imply that $1 < \alpha < 3$.
- For $\alpha > 3$ a Type IV singularity occurs.

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Example II: A singular cosmological evolution from Einstein frame

- Now we investigate how the scale factor behaves when it is conformally transformed, which in the Einstein frame reads,

$$\tilde{a}(\tilde{t}) = Ce^{\frac{f_0(\tilde{t}-\tilde{t}_s)^{1+\alpha}}{1+\alpha}}, \tag{21}$$

so the conformally transformed scale factor in the Jordan frame reads,

$$a(\tilde{t}) = a_0 e^{\left(\frac{3f_0(\tilde{t}-\tilde{t}_s)^{\alpha+1} \pm 2\sqrt{-3\alpha f_0}(\tilde{t}-\tilde{t}_s)^{\frac{\alpha+1}{2}}}{3(\alpha+1)}\right)}. \tag{22}$$

Example II: A singular cosmological evolution from Einstein frame

The later scale factor dictates the following singularity patter for the cosmological evolution, depending on the values of the parameter α,

- For $\alpha < -1$, a Type I or no singularity occurs.
- For $-1 < \alpha < 1$, a Type III singularity occurs.
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Singularity in Einstein Frame	**Singularity in Jordan Frame**
Type I	Type I or no singularity
Type III	Type III
Type II	Type III
Type IV	Type IV or Type II

Table: Correspondence for finite time singularities in the Einstein and Jordan frames, for the cosmological evolution $\tilde{H}(\tilde{t}) = f_0(\tilde{t} - \tilde{t}_s)^{\alpha}$ in the Einstein frame.

Unimodular $F(R)$ gravity - some properties

- Apart from the standard $F(R)$ gravity approach, there is an interesting theory which is called Unimodular $F(R)$ gravity.
- In this theory, the determinant of the metric is fixed.
- Unimodular gravity offers an interesting and conceptually simple proposal for the cosmological constant problem, since it is a particular case of general relativity, so no new physical assumptions are required.
- The cosmological constant originates from the trace-free part of the Einstein field equations, so the cosmological constant appears in the theory without adding it by hand, and more importantly it is not related to the vacuum expectation value of some matter field.

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- The problem of late-time acceleration can be addressed, in which case the full metric is decomposed in two parts, the unimodular metric and a scalar field.
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- Due to the interesting properties of this theory, we will also study if singularities change from one frame to another.

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Unimodular $F(R)$ gravity (Jordan frame)

- The action of unimodular $F(R)$ gravity reads

Unimodular $F(R)$ gravity action

$$S = \int d^4x \left\{ \sqrt{-g} \left(F(R) - \lambda \right) + \lambda \right\} , \tag{23}$$

where λ is a Lagrange multiplier function.

- If we do a variation with respect to λ, we obtain the unimodular constraint,

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$$\sqrt{-g} = 1 . \tag{24}$$

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Unimodular $F(R)$ gravity and a minimally coupled scalar tensor theory

- We can follow the same procedure as before and demonstrate that after a conformal transformation, there is an equivalent minimally scalar-tensor theory related to Unimodular $F(R)$ gravity, given by

Unimodular minimally coupled scalar-tensor theory (Einstein frame)

$$S = \int d^4x \left\{ \sqrt{-\tilde{g}} \left(\frac{\tilde{R}}{2\kappa^2} - \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) - \lambda h(\phi) \right) + \lambda \right\}, \quad (25)$$

where $h(\phi)$ is another potential.

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FRW Transformations between unimodular frames

- FRW metric does not satisfy the unimodular constraint so one needs to introduce a new coordinate $\tau = a(t)^3 dt$, which makes that the resulting metric

$$ds^2 = a\left(t\left(\tau\right)\right)^{-6} d\tau^2 - a\left(t\left(\tau\right)\right)^2 \left(dx^2 + dy^2 + dz^2\right), \quad (26)$$

  satisfies the unimodular condition.
- The unimodular constraint is affected by the conformal transformation. Indeed, we have $\sqrt{-\tilde{g}} = e^{2\sigma}$.
- In the Einstein frame, the new time coordinate needs to also satisfy the unimodular constraint so $d\tilde{\tau} = \tilde{a}(\tilde{t})^3 e^{2\sigma(\tilde{t})} d\tilde{t}$.
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Example: Unimodular $F(R) = R^{-n}$ singularity types in the Jordan frame

- Let us examine a model where we begin in the unimodular Jordan frame and conformally transform to the unimodular Einstein frame.
- We consider vacuum unimodular $F(R)$ gravity of the form $F(R) \sim R^{-n}$, in which case the scale factor behaves as follows*,

$$a(\tau) \sim (\tau_0 - \tau)^{\frac{1+3n+2n^2}{5+10n+6n^2}}, \tag{27}$$

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- A Big Rip singularity occurs in the τ coordinate, if the parameter n lies in the range $-1 < n < -1/2$.
- In terms of the original FRW cosmic time coordinate t, this means the scale factor evolves as,

$$a(t) \sim t^{\frac{(2n+1)(n+1)}{2+n}} . \tag{28}$$

  Thus a Big Rip singularity in the τ coordinate, appears in the t coordinate if $n < -2$ or $-1 < n < -1/2$.
- What happens in the corresponding Einstein frame scalar-tensor theory?

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Correspondence of $F(R)$ gravity singularities in different frames

We can conclude that:

First conclusion of this section

The standard and unimodular $F(R)$ gravity have both an equivalent minimally coupled scalar tensor theory when one conformally transforms the metric as $g_{\mu\nu} = e^{\sigma}\tilde{g}_{\mu\nu}$.

Second conclusion of this section

Cosmological time singularities might change from one frame to another for both the unimodular and standard case.

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Outline

What happens with acceleration/deceleration?

- We showed that time singularities might change from one frame to another. So, now we will ask the following important question:

Question

Is it possible to have a situation where we have acceleration (deceleration) of the universe in one frame and deceleration (acceleration) of the universe in the other?

Main objectives of this section

We shall investigate under which circumstances, an accelerating evolution in one frame may be transformed to a decelerating evolution in the other frame

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We shall investigate under which circumstances, an accelerating evolution in one frame may be transformed to a decelerating evolution in the other frame

What happens with acceleration/deceleration?

- We showed that time singularities might change from one frame to another. So, now we will ask the following important question:

Question

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How does acceleration/deceleration change from one frame to another?

- Reminder: the scale factors of the Jordan and Einstein frames are related as follows,

$$a(t(\tilde{t})) = e^{\frac{1}{2}\sqrt{\frac{2}{3}}\phi}\tilde{a}(\tilde{t}). \quad (30)$$

- The second derivative of the scale factor (acceleration) reads,

$$a'' = \left(\frac{1}{2}\sqrt{\frac{2}{3}}\ddot{\phi}\tilde{a} + \frac{1}{2}\sqrt{\frac{2}{3}}\dot{\phi}\dot{\tilde{a}} + \ddot{\tilde{a}}\right) e^{-\frac{1}{2}\sqrt{\frac{2}{3}}\phi}. \quad (31)$$

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Accelerating in the Einstein frame and decelerating in the Jordan frame

- We will study the case where we have acceleration in the Einstein frame ($\ddot{\tilde{a}} > 0$) and simultaneously deceleration in the Jordan frame ($a'' < 0$).
- In addition, the conditions $a' >$ and $\dot{\tilde{a}} > 0$ must hold true in order to have expansion in the both frames.
- In order for the above constraints to be satisfied, it suffices if the following conditions hold true,

$$A \equiv \frac{1}{2}\sqrt{\frac{2}{3}}\ddot{\phi}\tilde{a} + \frac{1}{2}\sqrt{\frac{2}{3}}\dot{\phi}\dot{\tilde{a}} + \ddot{\tilde{a}} > 0, \quad (32)$$

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- From the Einstein scalar-tensor eqs., it is easily to find that $\phi(\tilde{t}) = -\frac{2}{1+\alpha}\sqrt{-2f_0\alpha}\tilde{t}^{\frac{\alpha+1}{2}}$. Hence, $\alpha < -1$ and $f_0 > 0$.
- Therefore, it is easy to find that

$$a' = a f_0 \tilde{t}^{\alpha} \left(1 + \sqrt{\frac{-\alpha}{3f_0}}\tilde{t}^{\frac{-\alpha-1}{2}}\right) , \tag{34}$$

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- For the acceleration, we have

$$\ddot{\tilde{a}} = \tilde{a} f_0 \tilde{t}^{\alpha-1} \left(\alpha + f_0 \tilde{t}^{\alpha+1}\right) . \quad (36)$$

- This function may have different sign for the parameters chosen.
- According to our previous considerations, we are interested in the case that $\ddot{a} < 0$ and $A > 0$ or equivalently,

$$\frac{\alpha - 1}{2}\sqrt{\frac{-f_0\alpha}{3}} + \sqrt{\frac{-f_0\alpha}{3}} f_0 \tilde{t}^{\alpha+1} + f_0^2 \tilde{t}^{\frac{3\alpha+3}{2}} + f_0 \alpha \tilde{t}^{\frac{\alpha+1}{2}} > 0 , \quad (37)$$

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- It is expected that for some specific times, these inequalities will hold.
- Let us suppose that the first inequality is true for t_*, so we may put,

$$\tilde{t}_*^{\alpha+1} = m\frac{-\alpha}{f_0}, \tag{39}$$

where $0 < m < 1$ is some numerical parameter.

- Substituting this expression in the second inequality, we find,

$$B \equiv \frac{\alpha - 1}{2\sqrt{3}} - \frac{\alpha m}{\sqrt{3}} + m^{\frac{3}{2}}|\alpha| + m^{\frac{1}{2}}\alpha > 0. \tag{40}$$

- If the sum of the first two terms are positive, the above inequality will hold.

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- A detailed analysis of this inequality, imposes an additional restriction for the parameter, namely $\frac{1}{2} < m < 1$.
- By taking into account the negative values of α, the later inequality becomes

$$|\alpha| > \frac{1}{2\sqrt{3}m^{\frac{3}{2}} + 2m - 2\sqrt{3}m^{\frac{1}{2}} - 1}. \tag{41}$$

- The expression in the denominator is monotonically increasing function of m in the range $\frac{1}{2} < m < 1$, which crosses zero near the point $m \approx 0.8042$.
- All interesting values $0.8042 \lesssim m < 1$, for example $m = 0.9$ gives us $B \simeq 0.119 > 0$.

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- Therefore, all the inequalities can be true for some specific values of the parameters.

Result of the calculations:

We explicitly demonstrated for a specific model that for some parameters, simultaneously we might have an accelerating behaviour of the universe in the Einstein frame and a deceleration behaviour in the Jordan frame.

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Non-minimally curvature-coupled scalar theory

- Let us now consider the case where the Ricci scalar is non-minimally coupled to the scalar field:

Non-minimally curvature-coupled scalar theory (Jordan frame)

$$S = \int d^4x\sqrt{-g}\left[(1+f(\phi))\frac{R}{\kappa^2} - \frac{1}{2}\omega(\phi)\partial_\mu\phi\partial^\mu\phi - V(\phi)\right] . \quad (42)$$

- We shall refer to this non-minimally coupled frame as Jordan frame too (as we did with $F(R)$ gravity).

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UCL

Conformal transformation

- It can be shown that if we conformally transform our metric as

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$$g_{\mu\nu} = [1 + f(\phi)]^{-1}\tilde{g}_{\mu\nu} \ , \tag{43}$$

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Einstein frame corresponding to this theory

The corresponding minimally curvature-coupled scalar theory is given by

Minimally curvature-coupled scalar theory (Einstein frame)

$$S = \int d^4x\sqrt{-\tilde{g}}\left(\frac{\tilde{R}}{\kappa^2} - \frac{1}{2}W(\phi)\,\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - U(\phi)\right)\,, \qquad (44)$$

where the functions $W(\phi)$ and $U(\phi)$ are defined as follows,

$$W(\phi) = \frac{\omega(\phi)}{1+f(\phi)} + \frac{3}{\kappa^2(1+f(\phi))^2}\left(\frac{df(\phi)}{d\phi}\right)^2\,, \qquad (45)$$

$$U(\phi) = \frac{V(\phi)}{[1+f(\phi)]^2}\,. \qquad (46)$$

Flat FRW for the Einstein frame

For the later action and for a flat FRW metric, the cosmological equations can be written as follows (taking $\kappa^2 = 1$),

$$\tilde{H}^2 = \frac{\kappa^2}{6}\left(\frac{1}{2}W(\phi)\dot{\phi}^2 + U(\phi)\right) , \quad (47)$$

$$\dot{\tilde{H}} = -\frac{\kappa^2}{4}\left(W(\phi)\dot{\phi}^2\right) , \quad (48)$$

$$2W(\phi)\left[\ddot{\phi} + 3\tilde{H}\,\dot{\phi}\right] + \left[W_\phi\dot{\phi}^2 + 2U_\phi\right] = 0 , \quad (49)$$

where W_ϕ denotes partial differentiation with respect to ϕ.
Reminder: dots are differentiation with respect to $\tilde{t}$ and primes with respect to t.

How does acceleration change from one frame to another?

- We will ask the same question as we did before, How does acceleration change from one frame to another?
- Now: Jordan frame is non-minimally coupled-scalar theory and Einstein frame is minimally coupled-scalar theory.
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$$a(t) = [1 + f(\phi)]^{-1/2}\,\tilde{a}(\tilde{t})\,, \quad \frac{d\tilde{t}}{dt} = [1 + f(\phi)]^{1/2}\,. \qquad (50)$$

- By differentiating with respect to the time t, we find,

$$\frac{da(t)}{dt} \equiv a' = \dot{\tilde{a}} - \frac{1}{2}[1 + f(\phi)]^{-1} f_\phi \dot{\phi}\tilde{a}, \qquad (51)$$

- The second derivative is,

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- Let us assume that in the Jordan (Einstein) frame the Universe is decelerating (accelerating). To materialize such a scenario, the following inequalities need to hold true,

$$\begin{aligned} 2[1+f(\phi)]\ddot{\tilde{a}} + [1+f(\phi)]^{-1} f_\phi^2 \dot{\phi}^2 \tilde{a} - f_{\phi\phi}\dot{\phi}^2\tilde{a} - f_\phi \ddot{\phi}\tilde{a} - f_\phi(\dot{\phi}\tilde{a})^{\cdot} &< 0\,, \\ \ddot{\tilde{a}} &> 0\,. \end{aligned}$$

- Additionally, we need to impose that in each frame the Universe is expanding, hence

$$\dot{\tilde{a}} - \frac{1}{2}[1+f(\phi)]^{-1} f_\phi \dot{\phi}\tilde{a} > 0\,, \tag{53}$$

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Accelerating in the Einstein frame and decelerating in the Jordan frame

- Let us assume that in the Jordan (Einstein) frame the Universe is decelerating (accelerating). To materialize such a scenario, the following inequalities need to hold true,

$$\begin{aligned} 2[1+f(\phi)]\ddot{\tilde{a}} + [1+f(\phi)]^{-1} f_\phi^2 \dot{\phi}^2 \tilde{a} - f_{\phi\phi}\dot{\phi}^2\tilde{a} - f_\phi \ddot{\phi}\tilde{a} - f_\phi (\dot{\phi}\dot{\tilde{a}}) &< 0\,, \\ \ddot{\tilde{a}} &> 0\,. \end{aligned}$$

- Additionally, we need to impose that in each frame the Universe is expanding, hence

$$\dot{\tilde{a}} - \frac{1}{2}[1+f(\phi)]^{-1} f_\phi \dot{\phi}\tilde{a} > 0\,, \tag{53}$$

$$\dot{\tilde{a}} > 0\,. \tag{54}$$

Accelerating in the Einstein frame and decelerating in the Jordan frame

- Now, let us take the specific model where $\tilde{a}(\tilde{t}) = \tilde{a}_0 e^{\tilde{H}_0 \tilde{t}}$ (de-Sitter) and the coupling function is

$$f(\phi) = \frac{1 - \alpha\phi}{\alpha\phi} \,, \tag{55}$$

where α is a constant.

- This case is solution to the field eqs if

$$W(\phi) = 0 \rightarrow \quad \omega(\phi) = \frac{-3}{\alpha\phi^3} \,, \tag{56}$$

$$\text{and} \quad V(\phi) = \frac{6\tilde{H}_0^2}{\alpha^2\phi^2} \,, \tag{57}$$

$$\phi = \tilde{t} = \alpha^{\frac{-1}{3}} \left(\frac{3}{2} t\right)^{\frac{2}{3}} \,. \tag{58}$$

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Accelerating in the Einstein frame and decelerating in the Jordan frame

- It is easy to check that in both frames the universe is always expanding ($a' > 0$ and $\dot{\tilde{a}} > 0$)
- It is also easy to check that we always have $\ddot{\tilde{a}} > 0$ so we only need to check the big inequality showed before.
- Regard the big inequality, we have

$$\frac{\tilde{a}}{\alpha \tilde{t}^3}\left(2\tilde{H}_0^2\tilde{t}^2 + \tilde{H}_0\tilde{t} - 1\right) < 0 \, , \qquad (59)$$

and by also taking into account that $\tilde{t} > 0$, we find that this expression is satisfied for $0 < \tilde{t} < 1/(2\tilde{H}_0)$.

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Accelerating in the Einstein frame and decelerating in the Jordan frame

Result

For the non-minimally coupled-scalar tensor theory case, we also find that it is possible to have acceleration in one frame (Einstein) and deceleration in the other (Jordan).

Outline

Conclusions

- We presented how $F(R)$, unimodular $F(R)$ gravity and non-minimally coupled-scalar tensor theory (Jordan frames) are equivalent to specific minimally coupled scalar-tensor theories (Einstein frame) when we conformally transform the metric as $\tilde{g}_{\mu\nu} = e^{-\sigma} g_{\mu\nu}$.
- We presented how these two frames are mathematically related.
- We studied that cosmological time singularities might change from one frame to another.

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Conclusions

- We also showed that this behaviour is even more severe: There are situations in which we can have acceleration of the universe in one frame whereas we have deceleration of the universe in the other.
- With some specific models we explicitly showed this difference and how this scenario can be achieved for the Jordan frame and the Einstein frame.
- Therefore, we showed that even though these theories are mathematically equivalent, the physical interpretation of them might be different.

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