Solar System Tests in Modified Teleparallel gravity

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arXiv:2006.06750 in collaboration with Jackson Levi Said and M. Zubair

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Outline

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1. New perturbed spherically symmetric solutions
   - Theory and spherical symmetry
   - Perturbed spherically symmetric solutions
2. Phenomenology in $f(T,B)$ gravity
   - Potential and Basic ingredients
   - Solar System Tests
3. Conclusions

$f(T)$ gravity

- In analogy with $f(\mathring{R})$ gravity, one can consider

$f(T)$ gravity action

$$S_{f(T)} = \int f(T) e \, d^4x \, .$$

- The torsion scalar T depends on the first derivatives of the tetrads $\rightarrow$ **Second order theory**:

Not equivalency between $f(\mathring{R})$ and $f(T)$

Field equations of $f(T) \neq$ Field equations of $f(\mathring{R})$

- The reason is:

$$R = \mathring{R} + T - \frac{2}{e}\partial_\mu \left(eT^{\sigma}{}_{\sigma}{}^{\mu}\right) = 0 \Rightarrow \mathring{R} = -T + \frac{2}{e}\partial_\mu \left(eT^{\sigma}{}_{\sigma}{}^{\mu}\right)$$

$$\mathring{R} := -T + B$$

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- If $f(T,B) = f(T)$, one gets $f(T)$ gravity
- Other theories related to the boundary term such as $T + f(B)$ gravity.

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Spherical symmetry in $f(T, B)$ gravity

- Let us assume a spherically symmetric spacetime whose metric is

$$ds^2 = \mathcal{A}(r)dt^2 - \mathcal{B}(r)dr^2 - r^2 d\Omega^2\,,$$

where $\mathcal{A}(r)$ and $\mathcal{B}(r)$ are positive functions, which is reproduced by the off-diagonal tetrad

$$e^a{}_\mu = \begin{pmatrix} \sqrt{\mathcal{A}} & 0 & 0 & 0 \\ 0 & \sqrt{\mathcal{B}}\sin(\theta)\cos(\phi) & r\cos(\theta)\cos(\phi) & -r\sin(\theta)\sin(\phi) \\ 0 & \sqrt{\mathcal{B}}\sin(\theta)\sin(\phi) & r\cos(\theta)\sin(\phi) & r\sin(\theta)\cos(\phi) \\ 0 & \sqrt{\mathcal{B}}\cos(\theta) & -r\sin(\theta) & 0 \end{pmatrix}.$$

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Spherical symmetry in $f(T,B)$ gravity

For this tetrad, the $f(T,B)$ field equations yield

$$\begin{aligned}\frac{1}{2}\kappa^2\rho \quad = \quad & \frac{1}{4}f+\frac{r\mathcal{B}(\sqrt{\mathcal{B}}-1)\mathcal{A}'+\mathcal{A}(r\mathcal{B}'+2\mathcal{B}^{3/2}-2\mathcal{B})}{2r^2\mathcal{A}\mathcal{B}^2}f_T-\frac{r\mathcal{B}'f_B'-4\mathcal{B}^{3/2}\left(f_B'+f_T'\right)+4\mathcal{B}f_T'}{4r\mathcal{B}^2}\,, \\ & +\frac{r^2\mathcal{B}\mathcal{A}'^2+r\mathcal{A}\Big[r\mathcal{A}'\mathcal{B}'+4\mathcal{B}^{3/2}\mathcal{A}'-2\mathcal{B}(r\mathcal{A}''+4\mathcal{A}')\Big]+4\mathcal{A}^2(r\mathcal{B}'+2\mathcal{B}^{3/2}-2\mathcal{B})}{8r^2\mathcal{A}^2\mathcal{B}^2}f_B \\ & +\frac{f_B''}{2\mathcal{B}}\,,\end{aligned}$$

$$\begin{aligned}\frac{1}{2}\kappa^2 p_r \quad = \quad & -\frac{1}{4}f-\frac{r(\sqrt{\mathcal{B}}-2)\mathcal{A}'+2\mathcal{A}(\sqrt{\mathcal{B}}-1)}{2r^2\mathcal{A}\mathcal{B}}f_T-\frac{r\mathcal{A}'+4\mathcal{A}}{4r\mathcal{A}\mathcal{B}}f_B' \\ & +\frac{-r^2\mathcal{B}\mathcal{A}'^2+r\mathcal{A}\left[-r\mathcal{A}'\mathcal{B}'-4\mathcal{B}^{3/2}\mathcal{A}'+2\mathcal{B}(r\mathcal{A}''+4\mathcal{A}')\right]-4\mathcal{A}^2\left(r\mathcal{B}'+2\mathcal{B}^{3/2}-2\mathcal{B}\right)}{8r^2\mathcal{A}^2\mathcal{B}^2}f_B\end{aligned}$$

where primes denote differentiation with respect to the radial coordinate, then, $f_T'=f_{TT}T'+f_{TB}B'$ and $f_B'=f_{BB}B'+f_{TB}T'$

Perturbations around Schwarzschild

- Finding new exact solutions is difficult in $f(T,B)$, there have been some attempts but none of them have found physically interesting solutions so far.[†]
- Let us then assume that the background is described by the Schwarzschild geometry and the perturbed coefficients are first order corrections to this spacetime, namely,

$$\mathcal{A}(r) = 1 - \frac{2M}{r} + \epsilon\, a(r)\,,$$
$$\mathcal{B}(r) = \left(1 - \frac{2M}{r}\right)^{-1} + \epsilon\, b(r)\,.$$

Here $\epsilon \ll 1$ is a small tracking parameter.

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Model studied

- To include different power-law forms of the Lagrangian, let us assume the following combination of power-law terms

$$f(T,B) = T + \frac{1}{2}\epsilon\left(\alpha T^q + \beta B^m + \gamma B^s T^w + \zeta(\xi T + \chi B)^u\right) ,$$

where $\alpha, \beta, \gamma, \zeta, q, m, s, w$ and u are constants.

- The case $\beta = \gamma = \zeta = 0$ (power-law $f(T)$) was studied before and presented in the previous talk by Christian Pfeifer[‡].

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Perturbations around Schwarzschild

- By replacing f and taking the perturbed metric, we found six different solutions depending on the parameters. We split them into two cases:
  1. Case 1: ($\zeta = 0$, $q = m = 2$)
     a. $f(T,B) = T + \frac{1}{2}\epsilon\left(\alpha T^2 + \beta B^2 + \gamma B^s T^{1-s}\right)$, meaning $w = -s + 1$
     b. $f(T,B) = T + \frac{1}{2}\epsilon\left(\alpha T^2 + \beta B^2 + \gamma BT\right)$, meaning $w = s = 1$
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     b. $f(T,B) = T + \frac{1}{2}\epsilon\zeta(\xi T + \chi B)^4$, meaning $u = 4$
- In this talk, I will only focus on Case 1.

Perturbed solutions

- The Cases 1a-1d have the following solutions:

$$\begin{aligned}
\mathcal{A}(r) &= \mu^2 - \frac{1}{(\mu^2-1)^2 r^2}\Big[3\beta\mu^7 - \frac{1}{2}(\alpha+13\beta)\mu^6 - 4\beta\mu^5 + \frac{1}{2}(15\alpha+43\beta)\mu^4 \\
&\quad -\frac{2}{3}(32\alpha+35\beta)\mu^3 + \frac{1}{2}(33\alpha+13\beta)\mu^2 + 4\beta\mu - \frac{1}{6}(13\alpha+\beta) - \frac{\beta}{\mu} \\
&\quad -2(\alpha+\beta)\left(1-3\mu^2\right)\log\mu\Big] + \epsilon\,\tilde{a}_\gamma(r)\,, \\
\mathcal{B}(r) &= \mu^{-2} + \frac{\epsilon}{r^2(\mu^2-1)}\Big[\frac{1}{2}(25\alpha+37\beta) - 4(\alpha+2\beta)\mu - \frac{2(16\alpha+13\beta)}{3\mu} \\
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\end{aligned}$$

where $\mu^2 = 1 - 2M/r$.

- $\tilde{a}_\gamma(r)$ and $\tilde{b}_\gamma(r)$ depend on the model. All the solutions are asymptotically flat.
- Note that we have set the integrations constants in such a way that one gets the correct limit $a_1 = 0, b_1 = 0$ up to $1/r$

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Outline

Potential

- Similarly as Christian's talk, one gets the potential Eq:

$$\dot{r}^2 + 2V(r) = 0\,,$$

where we have defined the potential as

$$V(r) = -\frac{1}{2}\mathcal{B}^{-1}\Big(\frac{k^2}{\mathcal{A}} - \frac{h^2}{r^2} - \sigma\Big)\,.$$

- By replacing the metric functions and expanding up to first order in ϵ, we get that the potential becomes

$$V(r) = -\frac{1}{2}k^2 + \frac{1}{2}\left(1-\frac{2M}{r}\right)\left(\frac{h^2}{r^2}+\sigma\right)$$
$$+\frac{\epsilon}{2}\left[k^2\left(\frac{a(r)}{1-\frac{2M}{r}} + b(r)\left(1-\frac{2M}{r}\right)\right) - b(r)\left(\sigma+\frac{h^2}{r^2}\right)\left(1-\frac{2M}{r}\right)^2\right]\,.$$

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Photon sphere and Shadow of the black hole

- For circular photon orbits ($\sigma = 0$) we must have that the potential and its derivatives vanish, i.e., $V = V' = 0$.
- We need to expand the radial (circular) coordinate $r_c = r_0 + \epsilon\, r_1$, energy $k = k_0 + \epsilon\, k_1$ and angular momentum $h = h_0 + \epsilon\, h_1$.
- Then, one needs to solve order by order into the conditions $V = V' = 0$.
- After doing all this procedure, one gets the photon sphere (photons are forced to travel in a orbit) which is related to the shadow of the black hole. For each solutions, we have

$$r = r_{\rm GR} + \epsilon\, r_\epsilon = 3M + \epsilon\Big(\frac{0.141338\alpha}{M} + \frac{0.038204\beta}{M} + r_\gamma\Big)\,,$$

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- the shadow of the black hole predicted by power-law $f(T,B)$ will be enlarged or reduced for each solution.
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Perihelion shift

- One can rewrite the potential equation as

$$\frac{1}{2}\frac{\dot{r}^2}{\dot{\phi}^2}+\frac{1}{\dot{\phi}^2}V(r)=\frac{1}{2}\left(\frac{dr}{d\phi}\right)^2+\frac{r^4}{h^2}V(r)=0\,.$$

- One gets that the perihelion shifts can be defined as

$$\Delta\phi=2\pi\left(\frac{h}{r_c^2\sqrt{V''(r_c)}}-1\right)\,.$$

- For massive objects: we consider the potential V,V',V''. We evaluate the equations $V(r_c)=0$ and $V'(r_c)=0$ with $h=h_0+\epsilon\,h_1$ and $k=k_0+\epsilon\,k_1$. And then we replace this in the above expressions up to first order in ϵ.

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- For all the perturbed solutions found in the previous section, we find that the perihelion shift is given by $(q = M/r_c)$

$$\begin{aligned}\Delta\phi &= \Delta\phi_{\rm GR} + \epsilon\,\Delta\phi_\epsilon \\ &\approx 6\pi q + 27\pi q^2 + 135\pi q^3 + \mathcal{O}(q^4) + \\ &\quad \epsilon\,\pi\Big(\frac{12\beta q}{r_c^2} + \frac{8q^2(\alpha+10\beta)}{r_c^2} + \frac{q^3(194\alpha+1139\beta)}{2r_c^2} + \gamma\Delta\phi_\gamma\Big)\,,\end{aligned}$$

where $\Delta\phi_\gamma$ is:

- **Case 1a:** $\gamma = 0$ (If $\alpha, \beta > 0$, $\Delta\phi$ is bigger)
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Deflection of light

- We can define the minimal distance r_0 using $\dot{r}(r_0) = 0$ ($V(r_0) = 0$) and then using the potential one gets that the deflection of light from a radius r_0 to r is

$$\frac{d\phi}{dr} = \frac{\dot{\phi}}{\dot{r}} = \pm \frac{h}{r^2\sqrt{-2V(r)}} \Rightarrow \phi(r) = \pm \int_{r_0}^{r} d\bar{r} \frac{\mathcal{B}(\bar{r})^{1/2}}{\bar{r}^2} \left(\frac{\mathcal{A}(r_0)}{r_0^2 \mathcal{A}(\bar{r})} - \frac{1}{\bar{r}^2} \right)^{-1/2}.$$

- Now, we need to replace the metric and expand up to first order in ϵ.
- After this, we assume that $r, r_0 \gg 1$ and we consider only the leading term for each constant contribution.
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Deflection of light

- The deflection angle for all the models can be expressed as

$$\vartheta = \vartheta_{\rm GR} + \epsilon\,\vartheta_\epsilon \approx \frac{4M}{r_0} + \frac{M^2}{r_0^2}\left(\frac{15\pi}{4} - 4\right) + \frac{M^3}{r_0^3}\left(\frac{244-45\pi}{6}\right) + \epsilon\left[\frac{4M^3}{15 r_0^5}(16\alpha+\beta) + \gamma\vartheta_\gamma\right],$$

where ϑ_γ is:

- **Case 1a:** $\gamma = 0$, **Case 1b:** $\vartheta_\gamma \approx \frac{34M^3}{15r_0^5}$, **Case 1c:** $\vartheta_\gamma \approx \frac{4M^4}{rr_0^7}$, **Case 1d:** $\vartheta_\gamma \approx \frac{2M^4}{rr_0^7}$
- For all the solutions, if the constants are positive, ϑ is enlarged.

Deflection of light

- The deflection angle for all the models can be expressed as

$$\vartheta = \vartheta_{\rm GR} + \epsilon\,\vartheta_\epsilon \approx \frac{4M}{r_0} + \frac{M^2}{r_0^2}\left(\frac{15\pi}{4} - 4\right) + \frac{M^3}{r_0^3}\left(\frac{244-45\pi}{6}\right) + \epsilon\left[\frac{4M^3}{15r_0^5}(16\alpha+\beta) + \gamma\vartheta_\gamma\right],$$

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Shapiro Delay

- The Shapiro effect represents the time correction for the round trip of a radar signal that passes near a massive object in the presence of gravity
- To find this, one needs to calculate the time required for a radial signal from two different points r_0 to r, which gives

$$t(r,r_0)=\int_{r_0}^{r} d\bar{r}\sqrt{-2V(\bar{r})}=\int_{r_0}^{r} d\bar{r}\Big[\left(1-\frac{r_0^2\mathcal{A}(r_0)}{\bar{r}^2\mathcal{A}(\bar{r})}\right)\frac{\mathcal{A}(\bar{r})}{\mathcal{B}(\bar{r})}\Big]^{-1/2}.$$

- Then, we expand up to first order in ϵ and then we assumed $(r,r_0\gg 1)$
- The so-called retardation of light (or Shapiro delay) is then defined as

$$t_{\text{Shapiro}}(r,r_0)=t(r,r_0)-\sqrt{r^2-r_0^2}\,,$$

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- The time retardation for all the solutions can be written as,

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where $t_{\rm S,\gamma}$ is:

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Cassini experiment

- It was found that the fractional frequency shift y of a system composed of Earth-spacecraft-Earth (in a weak field limit) is given by

$$y = 2\frac{v_{\rm Cassini}l_{\rm Earth} + v_{\rm Earth}l_{\rm Cassini}}{l_{\rm Earth} + l_{\rm Cassini}}\vartheta\,,$$

  where ϑ is the deflection angle for the light, $l_{\rm Earth}, l_{\rm Cassini}$ are the distances from the Earth to the Sun and the Cassini spacecraft to the Sun, and $v_{\rm Earth}, v_{\rm Cassini}$ are the transverse velocities of the of the Earth and the Cassini spacecraft, respectively.

- Then, for the general form of y for each model can be written as

$$y = y_{\rm GR} + \epsilon\, y_\epsilon \approx \frac{8M}{r_0}v_{\rm Earth} + \epsilon\Big(\frac{8M^3(16\alpha+\beta)}{15r_0^5} + \gamma y_\gamma\Big)v_{\rm Earth}\,.$$

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Gravitational redshift

- Let us suppose that light is propagating at different heights r_1 and r_2 ($r_1 < r_2$). Then, the gravitational redshift is given by

$$z \equiv \frac{\nu_2}{\nu_1} - 1 = \sqrt{\frac{\mathcal{A}(r_2)}{\mathcal{A}(r_1)}} - 1\,,$$

where ν_1 and ν_2 are the frequencies measured from r_1 and r_2 respectively

- Following the same idea as before, we found

$$\left(\frac{\nu_2}{\nu_1}\right) \approx \left(\frac{\nu_2}{\nu_1}\right)_{\rm GR} + \epsilon\left[\frac{2}{5}M^3\alpha\left(r_1^{-5} - r_2^{-5}\right) + \beta M^2\left(r_1^{-4} - r_2^{-4}\right) + \gamma\left(\frac{\nu_2}{\nu_1}\right)_\gamma\right],$$

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Constraining the models

- For example, for the perihelion shift of Mercury, for GR the computation gives

$$\Delta\phi_{\text{GR,Mercury}} \approx 0.1033\,\text{"/cycles} \approx 42,84\,\text{"/cen}\,.$$

- The observed value is $42,98 \pm 0.040$ "/cen, so $\Delta\phi_\epsilon$ must lie between the error bars:

$$\Delta\phi_{\epsilon,\max} \approx 0.18\,\text{"/cen}\,.$$

- For example, for the Case 1a, one gets that the maximum value that the constants could be are

$$\Big|\alpha + 5.65\times 10^7\beta\Big|_{\max} \approx 3.65\times 10^{20}\,\text{km}^2\,.$$

- We did the same for all the Solar System tests to find the maximum value for the constants (see more details about what "data" we used in our paper)

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Summary of results - Max. values

Model	Perihelion shift	Deflection Light	Cassini	Shapiro delay	Grav. redshift
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Case 1b	$\|-\gamma\|\lesssim 10^{13}$	$\|\gamma\|\lesssim 10^{20}$	$\|\gamma\|\lesssim 10^{23}$	$\|-\gamma\|\lesssim 10^{18}$	$\|\gamma\|\lesssim 10^{13}$
Case 1c	$\|-\gamma\|\lesssim 10^{42}$	$\|\gamma\|\lesssim 10^{39}$	$\|\gamma\|\lesssim 10^{43}$	$\|\gamma\|\lesssim 10^{40}$	$\|-\gamma\|\lesssim 10^{38}$
Case 1d	$\|\gamma\|\lesssim 10^{43}$	$\|\gamma\|\lesssim 10^{39}$	$\|\gamma\|\lesssim 10^{43}$	$\|\gamma\|\lesssim 10^{41}$	$\|-\gamma\|\lesssim 10^{39}$

Table: Constrains for the different solutions with different Solar System tests only considering the order of magnitudes of the maximum values of the parameters. The values have dimensions of km^2, km^4 depending on the solutions, but we have omitted them here in order to safe space. For each case, we have rounded the numbers to only show their order of magnitude. Cases 1b-1d also contain the same α and β contributions from Case 1a, but we have omitted them for simplicity to only show the order of magnitude in γ. These contributions should also appear in Cases 1b-1d in the same way in Case 1a.

- Even though the numbers in the table look large, they are not dimensionless quantity. Then, it may be made arbitrarily large or small by a simple change of units.
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Outline

Conclusions

- We have found 6 new spherically symmetric solutions using perturbation methods around a Schwarzschild background in $f(T, B)$ gravity.
- We computed 6 different Solar System tests for all the models: photon sphere, perihelion shift, deflection of light, Shapiro delay, Cassini experiment and Grav redshift.
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