Spherical symmetry and black holes in teleparallel gravity

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Metric-Affine Frameworks for Gravity 2022, 28/Jun/2022

Based mainly on JCAP **01** (2022) no.01, 037 and our review arXiv:2106.13793

Outline

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- The torsion tensor is generally non-vanishing, and transforms covariantly under both diffeomorphisms and local Lorentz transformations (by a simultaneous transformation in the spin connection).

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- The Ricci scalar computed from the Levi-Civita connection $\overset{\circ}{R}$ differs from the scalar torsion T by a boundary term B.

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Equivalence between field equations

The field equations arising from $S_{\rm TEGR}$ are equivalent to the Einstein field equations.

Theories with scalar torsion and boundary term - $f(T, B)$

- One interesting modified TG theory is when one considers[1]

[1] S. Bahamonde, C. G. Böhmer and M. Wright, Phys. Rev. D **92** (2015) no.10, 104042

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- If $f(T,B) = f(T)$, one gets $f(T)$ gravity
- There has been quite a lot of study about this theory in the last years.

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Field equations and spherical symmetry

- The separate tetrad and spin connection variations produce the field equations

$$W_{(\mu\nu)} = \kappa^2 \Theta_{\mu\nu} \,,$$
$$W_{[\mu\nu]} = \Big[(\partial_\rho f_B) + (\partial_\rho f_T)\Big] S_{[\mu}{}^{\rho}{}_{\nu]} \propto T^{\rho}{}_{[\mu\nu}\partial_{\rho]}(f_T + f_B) = 0 \,.$$

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- The three equations of motion obtained are not fully independent of each other.
- The antisymmetric field equation coincides with the spin connection equation.

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$$e^{A}{}_{\mu} \longrightarrow e^{a}{}_{\mu} - e^{a}{}_{\nu}\partial_{\mu}\zeta^{\nu} - \zeta^{\nu}\partial_{\nu}e^{a}{}_{\mu}\,,$$

which for the equations of motion $W^{\mu\nu} = 0$ with

$$\kappa^2 \frac{\delta \mathcal{S}}{\delta e^{a}{}_{\mu}} \equiv e W^{\mu\nu} e^{b}{}_{\nu}\eta_{ab}\,,$$

implies the following simple relation:

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- Thus, in $f(T,B)$ gravity, if the antisymmetric part of equations is satisfied, then the covariant divergence of equations of motion vanishes identically.

Field equations and spherical symmetry

Killing eqs:

$$\mathcal{L}_{Z_\zeta} e^A{}_\mu = -\lambda_\zeta^A{}_B e^B{}_\mu\,, \quad \mathcal{L}_{Z_\zeta} \omega^A{}_{B\mu} = \partial_\mu \lambda_\zeta^A{}_B + \omega^A{}_{C\mu} \lambda_\zeta^C{}_B - \omega^C{}_{B\mu} \lambda_\zeta^A{}_C\,.$$

[2] M. Hohmann, L. Järv, M. Krššák and C. Pfeifer, Phys. Rev. D **100** (2019) no.8, 084002

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By solving these eqs in the Weitzenböck gauge we find[2]

$$e^A{}_\nu = \begin{pmatrix} C_1 & C_2 & 0 & 0 \\ C_3 \sin\vartheta\cos\varphi & C_4\sin\vartheta\cos\varphi & C_5\cos\vartheta\cos\varphi - C_6\sin\varphi & -\sin\vartheta(C_5\sin\varphi + C_6\cos\vartheta\cos\varphi) \\ C_3\sin\vartheta\sin\varphi & C_4\sin\vartheta\sin\varphi & C_5\cos\vartheta\sin\varphi + C_6\cos\varphi & \sin\vartheta(C_5\cos\varphi - C_6\cos\vartheta\sin\varphi) \\ C_3\cos\vartheta & C_4\cos\vartheta & -C_5\sin\vartheta & C_6\sin^2\vartheta \end{pmatrix},$$

where the six free functions $C_I = C_I(t,r)$ $(I = 1,..6)$ can depend on time and the radial coordinate

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Solving antisymmetric field equations

There are two different tetrads which solve the antisymmetric field equation and they have the same metric[3]

$$e_{(1)}{}^{A}{}_{\mu} = \begin{pmatrix} \mathcal{A}(r) & 0 & 0 & 0 \\ 0 & \mathcal{B}(r)\sin\vartheta\cos\varphi & \xi r\cos\vartheta\cos\varphi & -r\xi\sin\vartheta\sin\varphi \\ 0 & \mathcal{B}(r)\sin\vartheta\sin\varphi & \xi r\cos\vartheta\sin\varphi & \xi r\sin\vartheta\cos\varphi \\ 0 & \mathcal{B}(r)\cos\vartheta & -r\xi\sin\vartheta & 0 \end{pmatrix}, \quad \xi = \pm 1\,,$$

$$e_{(2)}{}^{A}{}_{\mu} = \begin{pmatrix} 0 & i\mathcal{B}(r) & 0 & 0 \\ i\mathcal{A}(r)\sin\vartheta\cos\varphi & 0 & -r\sin\varphi & -r\sin\vartheta\cos\vartheta\cos\varphi \\ i\mathcal{A}(r)\sin\vartheta\sin\varphi & 0 & \chi r\cos\varphi & -r\sin\vartheta\cos\vartheta\sin\varphi \\ i\mathcal{A}(r)\cos\vartheta & 0 & 0 & r\sin^2\vartheta \end{pmatrix},$$

$$\mathrm{d}s^2 = \mathcal{A}(r)^2\mathrm{d}t^2 - \mathcal{B}(r)^2\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2\,.$$

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$$e^A_{(1)\,\mu} = \begin{pmatrix} \mathcal{A}(r) & 0 & 0 & 0 \\ 0 & \mathcal{B}(r)\sin\vartheta\cos\varphi & \xi r\cos\vartheta\cos\varphi & -r\xi\sin\vartheta\sin\varphi \\ 0 & \mathcal{B}(r)\sin\vartheta\sin\varphi & \xi r\cos\vartheta\sin\varphi & \xi r\sin\vartheta\cos\varphi \\ 0 & \mathcal{B}(r)\cos\vartheta & -r\xi\sin\vartheta & 0 \end{pmatrix}, \quad \xi = \pm 1\,,$$

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The phenomenology of these two tetrads will be different! We found exact solutions for different f.

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Some remarks about the tetrads

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- Minwkoski: If $\xi=+1$, the $T\to 0$ for $\mathcal{A}(r),\mathcal{B}(r)\to 1$ whereas one needs to take a "weird" Minwkoski limit $\mathcal{A}(r),\mathcal{B}(r)\to -1$ to get $T\to 0$ for $\xi=-1$.

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- For the complex tetrad, all the quantities (torsion tensor, scalars, etc) are real.

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- For the complex tetrad, all the quantities (torsion tensor, scalars, etc) are real.
- Since we couple matter with the metric, these imaginary terms are not seen, so nothing is wrong with it.

Solutions for the complex tetrad

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- One can use the Bianchi identities to find that for any form of f, the metric functions MUST respect

$$\mathcal{B}(r) = \pm \frac{\sqrt{-r^2\mathcal{A}(r)\mathcal{A}''(r) - r^2\mathcal{A}(r)'^2 + \mathcal{A}(r)^2}}{\mathcal{A}(r)}\,.$$

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- This condition reduces the number of unknown from three ($\mathcal{A},\mathcal{B}$ and $f(T)$) to two ($\mathcal{A}(r)$ and f) which can be obtained by using the rr field eq:

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- Thus, one can choose a f and solve the above equation or one can choose $\mathcal{A}(r)$ and find the f satisfying that choice.
- Using these two approaches, we found exact solutions.

Solutions for the complex tetrad - similar RN

- The first interesting black hole one is

$$\mathrm{d}s^2 = \Big(1 - \frac{2M}{r} + \frac{Q}{r^2}\Big)dt^2 - \Big(\frac{2Mr - Q - r^2}{2Q - r^2}\Big)^{-1}dr^2 - r^2 d\Omega^2\,,$$

which looks like Reissner–Nordström but it does not have $g_{tt} = -1/g_{rr}$.

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  which looks like Reissner–Nordström but it does not have $g_{tt}=-1/g_{rr}$.
- This solution has two event horizons and can have any sign for Q.
- The form of the theory is

$$f(T)=4f_0\,\frac{\Big(2\pm\sqrt{Q^2T^2-2QT+4}\Big)}{\Big(QT+2\pm\sqrt{Q^2T^2-2QT+4}\Big)\sqrt{8-2QT\pm4\sqrt{Q^2T^2-2QT+4}}}\,.$$

Solution for the complex tetrad - Born-Infield

- A quite well-studied theory inspired by Born-Infield electromagnetism is

$$f(T) = \lambda\left(\sqrt{1+\frac{2T}{\lambda}}-1\right),$$

with λ being the so-called Born-Infeld parameter. It is easy to notice that when $T/\lambda \ll 1$, one obtains $f(T) = T - T^2/(2\lambda) + \mathcal{O}(1/\lambda^2)$.

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- We found an exact black hole solution to this theory(JCAP **01** (2022) no.01, 0370)

$$\mathrm{d}s^2 = \frac{a_1^2}{r}\left[\sqrt{\lambda}(a_0\lambda + r) - 2\tan^{-1}\left(\frac{\sqrt{\lambda}r}{2}\right)\right]\mathrm{d}t^2 - \frac{\lambda^{5/2}r^5}{(4+r^2\lambda)^2}\left[\sqrt{\lambda}(a_0\lambda + r) - 2\tan^{-1}\left(\frac{\sqrt{\lambda}r}{2}\right)\right]^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2\,. \quad (1)$$

Solutions for the complex tetrad - Born-Infield

One can set $a_1^2 = 1/\sqrt{\lambda}$ to get asymptotically flatness. Further, if we choose $a_0 = -2M/\lambda$ and expands the metric up to $\mathcal{O}(1/\lambda^2)$, we find

$$
\begin{aligned}
ds^2 &= \Big[1 - \frac{2M}{r} + \frac{4}{\lambda r^2} - \frac{\pi}{\sqrt{\lambda} r}\Big] dt^2 \\
&\quad - \Big[1 - \frac{2M}{r} - \frac{16M}{\lambda r^3} + \frac{12}{\lambda r^2} - \frac{\pi}{\sqrt{\lambda} r}\Big]^{-1} dr^2 - r^2 d\Omega^2 + \mathcal{O}(1/\lambda^2)\,.
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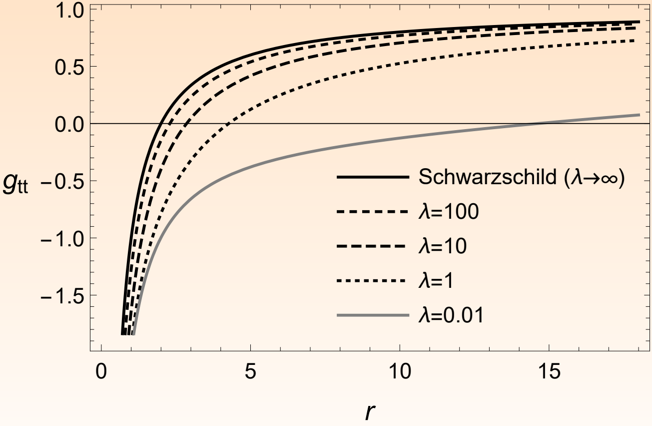
This is a generalization of a Schwarzschild black hole with one horizon $r_h = 2M + \frac{\pi}{\sqrt{\lambda}} - \frac{2M}{\lambda} + \mathcal{O}(1/\lambda^2)$.

Solutions for the complex tetrad - Born-Infield

We also checked numerically that there is only one horizon:



Theorem spherical symmetry - valid for the two tetrads

In $f(T)$ gravity

In $f(T)$ gravity, only for the case where the model is at most TEGR + Constant, the $\mathcal{A}(r)$ and $\mathcal{B}(r)$ take on the reciprocal of each other ($g_{tt} = -1/g_{rr}$). Moreover, the solution in this case is the Schwarzschild de Sitter solution.