Reviving Horndeski gravity using Teleparallel gravity

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Abstract

General Relativity (GR) is based on a manifold with curvature and zero torsion and on the contrary, Teleparallel gravity (TG) is a theory which assumes a **non-zero torsion with zero curvature**. It turns out that it is possible to write down a theory in Teleparallel gravity that **is equivalent to GR in terms of the field equations**. Even though these theories are equivalent in field equations, **their modifications are different**. Horndeski gravity which is based from GR **was highly constraint** from the recent gravitational waves observations due to $|c_g/c - 1| \gtrsim 10^{-15}$. We constructed an **analogue version of Horndenki gravity** which is based in Teleparallel gravity and showed that in this context, it is possible to **construct a theory satisfying** $c_T = c_g/c = 1$ without eliminating the coupling functions $G_5(\phi, X)$ and $G_4(\phi, X)$ that were highly constraint in standard Horndeski theory. Hence, in the Teleparallel approach, one is **able to restore these terms** creating an interesting way to revive Horndeski gravity.

1. What is Teleparallel gravity?

- It is constructed by the **Weitzenböck connection** $\Gamma^a{}_{\mu\nu}$ which is **curvatureless**, contains **torsion** and **satisfies the metric compatibility condition** $\nabla_\lambda g_{\mu\nu} = 0$.
- Our dynamical variables are the **tetrad fields** $h^a{}_\mu$ where Latin indices indicate tangents space coordinates and Greek indices correspond to spacetime coordinates. The metric can be reconstructed using the following relationship

$$g_{\mu\nu} = h^a{}_\mu h^b{}_\nu \eta_{ab}\,, \tag{1}$$

where η_{ab} is the Minkowski metric $(-+++)$.

- The field strength of the theory is the **torsion tensor** which is defined as the anti-symmetric part of the Weitzenböck connection, namely,

$$T^a{}_{\mu\nu} := 2\Gamma^a{}_{[\mu\nu]} = \partial_\mu h^a{}_\nu - \partial_\nu h^a{}_\mu + \omega^a{}_{b\mu} h^b{}_\nu - \omega^a{}_{b\nu} h^b{}_\mu\,, \tag{2}$$

where $\omega^a{}_{b\nu}$ is the spin connection. This quantity is generally non-vanishing and transforms covariantly under both diffeomorphisms and local Lorentz transformations.

- The torsion tensor can be **decomposed** as follows

$$T_{\lambda\mu\nu} = \frac{2}{3}(t_{\lambda\mu\nu} - t_{\lambda\nu\mu}) + \frac{1}{3}(g_{\lambda\mu}v_\nu - g_{\lambda\nu}v_\mu) + \epsilon_{\lambda\mu\nu\rho}a^\rho\,, \tag{3}$$

where

$$v_\mu = T^\lambda{}_{\lambda\mu}\,, \quad a_\mu = \frac{1}{6}\epsilon_{\mu\nu\sigma\rho}T^{\nu\sigma\rho}\,, \tag{4}$$

$$t_{\lambda\mu\nu} = \frac{1}{2}(T_{\lambda\mu\nu} + T_{\mu\lambda\nu}) + \frac{1}{6}(g_{\nu\lambda}v_\mu + g_{\nu\mu}v_\lambda) - \frac{1}{3}g_{\lambda\mu}v_\nu\,, \tag{5}$$

are three irreducible parts with respect to the local Lorentz group, known as the **vector, axial, and purely tensorial**, torsions, respectively.

- The most studied teleparallel model is the **Teleparallel equivalent of General Relativity** (TEGR) where the Lagrangian is assumed to take the form

$$\mathcal{L}_{\rm TEGR} = \frac{1}{4}T^\rho{}_{\mu\nu}T_\rho{}^{\mu\nu} + \frac{1}{2}T^\rho{}_{\mu\nu}T^{\nu\mu}{}_\rho - T^\lambda{}_{\lambda\mu}T_\nu{}^{\nu\mu} = \frac{3}{2}T_{\rm ax} + \frac{2}{3}T_{\rm ten} - \frac{2}{3}T_{\rm vec} \equiv T\,, \tag{6}$$

where we have defined **three invariants** $T_{\rm ten} = t_{\lambda\mu\nu}t^{\lambda\mu\nu}$, $T_{\rm ax} = a_\mu a^\mu$ and $T_{\rm vec} = v_\mu v^\mu$ and the **scalar torsion** T which is a linear combination of them.

- For a manifold with zero curvature satisfying the metric compatibility condition and non-zero torsion, one can get the following relationship

$$R = \mathring{R} + T - \frac{2}{h}\partial_\mu\left(hT^\sigma{}_\sigma{}^\mu\right) = 0 \quad \Rightarrow \quad \mathring{R} = -T + \frac{2}{h}\partial_\mu\left(hT^\sigma{}_\sigma{}^\mu\right) := -T + B\,. \tag{7}$$

Here, $\mathring{R}$ is the Ricci scalar as determined using the Levi-Civita connection, R is the Ricci scalar as calculated with the Weitzenböck connection which vanishes, and h is the determinant of the tetrad field, $h = \det\left(h^a{}_\mu\right) = \sqrt{-g}$. Thus, **the scalar torsion** T **differs only by a boundary term** B **with respect to** $\mathring{R}$ and then, since the TEGR Lagrangian is given by (6), the **TEGR field equations are the same as the Einstein field equations**.

- In analogy to the well-known $f(\mathring{R})$ gravity, one can consider $f(T)$ **gravity** in the TEGR framework which is constructed with the Lagrangian $\mathcal{L}_{f(T)} = f(T)$. Since the torsion scalar T only depends on the first derivatives of the tetrads, this theory is a **second-order theory**. $f(T)$ gravity does not differ from $f(\mathring{R})$ by a total derivative term, so that, these theories are no longer equivalent.
- Then, in general, **even though GR and TEGR are equivalent, when one starts modifying them, one gets different modified theories of gravity**.

2. Teleparallel Horndeski gravity

- We construct an **analogue** version of **Horndeski gravity theory** in the **Teleparallel** framework with the following **conditions**:
  1. The resulting field equations **must be at most second-order in terms of derivatives of the tetrad fields** (or equivalently second order in terms of metric tensor derivatives)
  2. The scalar invariants **should not be parity violating**.
  3. The field equations **must be covariant under local Lorentz transformations**.
  4. **Contractions of the torsion tensor can at most be quadratic**. Any number of contractions of the irreducible parts of the torsion tensor will result in second order field equations. This means that an infinite number of terms can be formed in Teleparallel gravity that gives rise to second-order field equations. However, it is unclear how physical such higher order contributions will be.
- The **most general Lagrangian** that has the above properties and gives second order field equations in **Minkowski space-time** is

$$L = c_1\phi + c_2X - c_3X\bar{\Box}\phi + c_4X\left[(\bar{\Box}\phi)^2 - \partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi\right] - \\ - c_5X\left[(\bar{\Box}\phi)^3 - 3(\bar{\Box}\phi)\,\partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi + 2\partial_\mu\partial_\nu\phi\partial^\nu\partial^\lambda\phi\partial_\lambda\partial^\mu\phi\right]\,, \tag{8}$$

where $X = -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi$ is the kinetic term, c_i are constants (that can be functions of ϕ and X to be more general) and $\bar{\Box} = \partial_\mu\partial^\mu$.

- In order to **introduce gravity**, one needs to replace quantities according to the table:

GR	Teleparallel
$\eta_{\mu\nu} \to g_{\mu\nu}$	$e^a{}_\mu \to h^a{}_\mu$
$\partial_\mu \to \mathring{\nabla}_\mu$	$\partial_\mu \to \mathcal{D}_\mu = \partial_\mu + h^c{}_\mu w^a{}_{bc}S^b_a$

Here $e^a{}_\mu$ represents tetrad in trivial frames and $S_c{}^b$ is a representation of the Lorentz generators. Hence, **standard Horndeski** is constructed by using the **covariant coupling prescription of GR** and **Teleparallel Horndeski** needs to be constructed by using the **Teleparallel coupling prescription**.

- **The Teleparallel prescription** $\mathcal{D}_\mu$, in the end, **coincides** with the **GR coupling prescription** $\mathring{\nabla}_\mu$ which is a **covariant derivative** with respect to the **Levi-Civita connection**. Then, the Teleparallel Lagrangians $\sum_{i=3}^5 \mathcal{L}$ **are identical** to the **last three terms** of the standard Horndeski gravity Lagrangian $\sum_{i=3}^5 \mathring{L}_i$.
- However, when one is considering Teleparallel gravity, $\mathcal{L}_2$ **would be different** to $\mathring{L}_2$ since there are more scalars that one can construct which satisfies the conditions.
- Taking **quadratic contractions** of the torsion tensor, **the most general Lagrangian** of Teleparallel gravity satisfying the conditions turns out to be $f(T_{\rm ax}, T_{\rm vec}, T_{\rm ten})$ (without a scalar field). If one **adds the scalar field**, one can construct the following **7 extra independent scalars**:

$$I_2 = v^\mu\phi_{;\mu}\,, \quad J_1 = a^\mu a^\nu\phi_{;\mu}\phi_{;\nu}\,, \quad J_3 = v_\sigma t^{\sigma\mu\nu}\phi_{;\mu}\phi_{;\nu}\,, \quad J_5 = t^{\sigma\mu\nu}t_\sigma{}^{\bar{\mu}}{}_\nu\phi_{;\mu}\phi_{;\bar{\mu}}\,, \tag{9}$$

$$J_6 = t^{\sigma\mu\nu}t_\sigma{}^{\bar{\mu}\bar{\nu}}\phi_{;\mu}\phi_{;\nu}\phi_{;\bar{\mu}}\phi_{;\bar{\nu}}\,, \quad J_8 = t^{\sigma\mu\nu}t_{\sigma\mu}{}^{\bar{\nu}}\phi_{;\nu}\phi_{;\bar{\nu}}\,, \quad J_{10} = \epsilon^\mu{}_{\nu\rho\sigma}a^\nu t^{\alpha\rho\sigma}\phi_{;\mu}\phi_{;\alpha}\,, \tag{10}$$

therefore, the **extra Lagrangian term** $\mathcal{L}_{\rm Tele} = G_{\rm Tele}(\phi, X, T, T_{\rm ax}, T_{\rm vec}, I_2, J_1, J_3, J_5, J_6, J_8, J_{10})$ is needed to be introduced in Teleparallel Horndeski.

- Considering all the possible terms, the final **Lagrangian of Teleparallel Horndeski** is a linear combination of the **standard Horndeski theory plus a new contribution**, namely,

$$\mathcal{L}_{\rm TeleDeski} = \sum_{i=2}^5 \mathcal{L}_i + \mathcal{L}_{\rm Tele} = {\rm Horndeski} + \mathcal{L}_{\rm Tele}\,, \tag{11}$$

where

$$\mathcal{L}_{\rm Tele} = G_{\rm Tele}(\phi, X, T, T_{\rm ax}, T_{\rm vec}, I_2, J_1, J_3, J_5, J_6, J_8, J_{10})\,,$$
$$\mathcal{L}_2 = G_2(\phi, X)\,, \quad \mathcal{L}_3 = G_3(\phi, X)\Box\phi\,,$$
$$\mathcal{L}_4 = G_4(\phi, X)\,(-T + B) + G_{4,X}(\phi, X)\left[(\Box\phi)^2 - \phi_{;\mu\nu}\phi^{;\mu\nu}\right]\,,$$
$$\mathcal{L}_5 = G_5(\phi, X)\mathcal{G}_{\mu\nu}\phi^{;\mu\nu} - \frac{1}{6}G_{5,X}(\phi, X)\left[(\Box\phi)^3 + 2\phi_{;\mu}{}^\nu\phi_{;\nu}{}^\alpha\phi_{;\alpha}{}^\mu - 3\phi_{;\mu\nu}\phi^{\mu\nu}(\Box\phi)\right].$$

Here, semicolon represents differentiation with respect to the Levi-Civita connection and $\Box\phi = \phi_{;\mu}{}^\mu$.

- In Horndeski gravity, $f(\mathring{R})$ does not appear directly since it is a **4th order theory**. In Teleparallel Horndeski, $f(T)$ appears in the Lagrangian since it is a **2nd order theory**.
- Teleparallel Horndeski **has a richer structure** since standard Horndeski is contained on it (setting $G_{\rm Tele} = 0$). See **Figure to see how the theories are connected**.

3. How are all the theories connected?

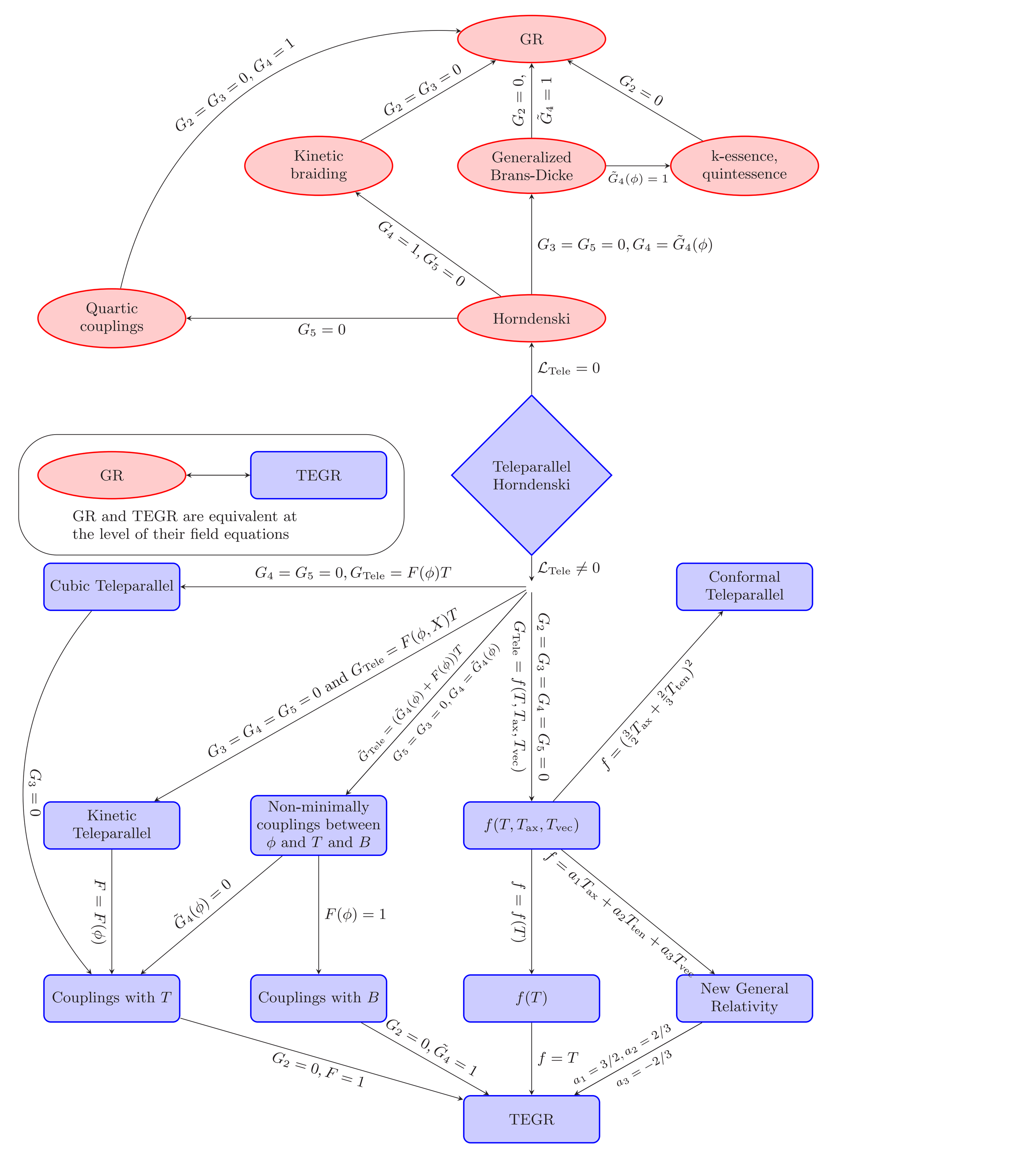

Figure: Relationship between Teleparallel Horndeski and various theories

4. Reviving Horndeski gravity using Teleparallel gravity

- By taking a **flat FLRW background** $ds^2 = -N(t)^2dt^2 + a(t)^2(dx^2 + dy^2 + dz^2)$ which is reproduced by $h^a{}_\mu = {\rm diag}(N(t), a(t), a(t), a(t))$ and then by performing **cosmological perturbations** $\bar{h}^a{}_\mu$, the perturbed tetrad becomes

$$\bar{h}^{(0)}{}_\mu = \delta^0_\mu\,(1 + \Psi) + a\delta^i_\mu\,(G_i + \partial_i F)\,, \tag{12}$$

$$\bar{h}^{(k)}{}_\mu = a\left[\delta^k_\mu\,(1 - \Phi) + \frac{1}{2}\delta^i_\mu\delta^{kj}\left(h_{ij} + \partial_i\partial_j B_s + \partial_j C_i + \partial_i C_j\right) + \delta^k_0\delta^j_\mu\partial_j\bar{F}\right]\,, \tag{13}$$

where G_i and C_i are vectorial perturbations, $F, \bar{F}, \psi, \Phi$ and B_s are the scalar perturbations and h_{ij} is the tensorial perturbation.

- If one considers **tensorial perturbations** only, it is possible to find that the **propagation of the gravitational waves for Teleparallel Horndeski** is given by

$$c_T^2 = \frac{G_4 - X(\ddot{\phi}G_{5,X} + G_{5,\phi}) - G_{\rm Tele,T}}{G_4 - 2XG_{4,X} - X(H\dot{\phi}G_{5,X} - G_{5,\phi}) + 2XG_{\rm Tele,J_8} + \frac{1}{2}XG_{\rm Tele,J_5} - G_{\rm Tele,T}}\,. \tag{14}$$

- For standard Horndeski ($G_{\rm Tele} = 0$), one finds that $c_T = 1$ only if $G_5(\phi, X) = {\rm constant}$ and $G_4(\phi, X) = G_4(\phi)$. This condition **highly constrains Horndeski gravity**.
- When the Teleparallel term is switch on, from (14) one finds that it is possible **to have a theory respecting** $c_T = 1$ with **non-trivial coupling functions** $G_5 = G_5(\phi)$ and also $G_4(\phi, X)$ depending on X too. The following **Lagrangian respecting** $c_T = 1$ is found:

$$\mathcal{L} = \tilde{G}_{\rm tele}(\phi, X, T, T_{\rm vec}, I_2) + \sum_{i=2}^4 \mathcal{L}_i + G_5(\phi)\mathcal{G}_{\mu\nu}\phi^{;\mu\nu}\,, \tag{15}$$

which revives $G_5(\phi)$ **and** $G_4(\phi, X)$. Therefore, one **can restore the coupling functions that were ruled out in Horndeski due to GW observations by considering Teleparallel Horndeski gravity**.