

# Black holes solutions in metric-affine gravity with dynamical torsion and nonmetricity

Sebastián Bahamonde

Postdoctoral Researcher at Tokyo Institute of Technology, Japan

Waseda University seminar, 22/March/2022

Based on JCAP **09** (2020), 057, Eur.Phys.J.C **81** (2021) 6, 495;

JCAP **01** (2022) no.01, 011; arXiv:2201.10532 (to appear in JCAP); Jointly with Jorge Gigante Valcarcel.



東京工業大学  
Tokyo Institute of Technology



# Outline

- 1 Introduction to Metric-affine gravity
  - Why modified gravity?
  - Basic geometrical quantities
  - Tetrads and spin connection
- 2 Trinity of gravity
  - Trinity of gravity: GR, TEGR and STEGR.
- 3 Metric-Affine gravity
  - Gauge formalism
  - Dynamics
- 4 MAG models with dynamical torsion and nonmetricity
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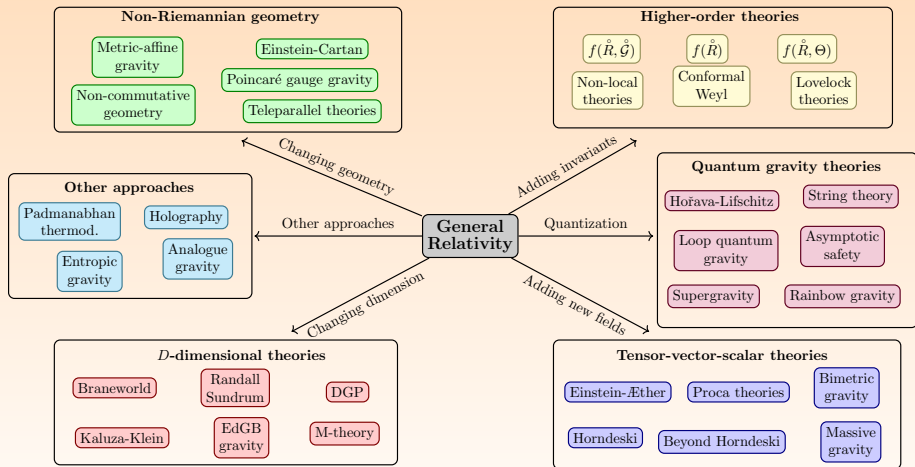


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- A good way to understand GR is to modify it;



# How to modify it?



**Figure:** Classification of theories of gravity. (S. Bahamonde et.al., “Teleparallel Gravity: From Theory to Cosmology,” [arXiv:2106.13793 [gr-qc]].)



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# Fundamental variables and characteristic tensors

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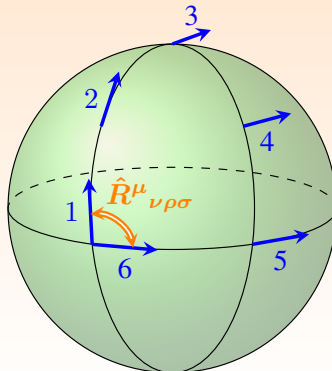
<b>Curvature</b>	$\tilde{R}^\mu{}_{\nu\rho\sigma} = \partial_\rho \tilde{\Gamma}^\mu{}_{\nu\sigma} - \partial_\sigma \tilde{\Gamma}^\mu{}_{\nu\rho} + \tilde{\Gamma}^\mu{}_{\tau\rho} \tilde{\Gamma}^\tau{}_{\nu\sigma} - \tilde{\Gamma}^\mu{}_{\tau\sigma} \tilde{\Gamma}^\tau{}_{\nu\rho}$
<b>Torsion</b>	$\tilde{T}^\mu{}_{\nu\rho} = \tilde{\Gamma}^\mu{}_{\rho\nu} - \tilde{\Gamma}^\mu{}_{\nu\rho}$
<b>Nonmetricity</b>	$\tilde{Q}_{\mu\nu\rho} = \tilde{\nabla}_\mu g_{\nu\rho} = \partial_\mu g_{\nu\rho} - \tilde{\Gamma}^\sigma{}_{\nu\mu} g_{\sigma\rho} - \tilde{\Gamma}^\sigma{}_{\rho\mu} g_{\nu\sigma}$



# What does curvature geometrically represent?

**Curvature tensor**  $\tilde{R}^\alpha{}_{\beta\mu\nu}$

Rotation experienced by a vector when it is parallel transported along a closed curve

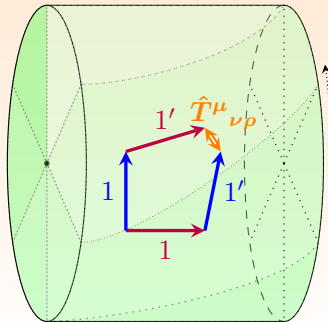




# What does torsion geometrically represent?

## Torsion tensor $\tilde{T}^{\alpha}_{\mu\nu}$

non-closure of the parallelogram formed when two infinitesimal vectors are parallel transported along each other.

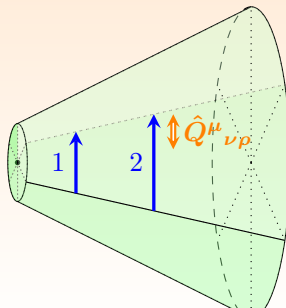




# What does non-metricity geometrically represent?

## Non-metricity tensor $\tilde{Q}_{\alpha\mu\nu}$

measures how much the length and angle of vectors change as we parallel transport them, so in metric spaces the length of vectors is conserve





## Some important special cases of metric-affine geometries

- **Riemann-Cartan geometry** ( $\tilde{Q}_{\alpha\mu\nu} = 0$ ): If non-metricity vanishes, the metric satisfies the metric-compatibility condition  $\tilde{\nabla}_\mu g_{\alpha\beta} = 0$ . Poincaré gravity assumes this geometry.



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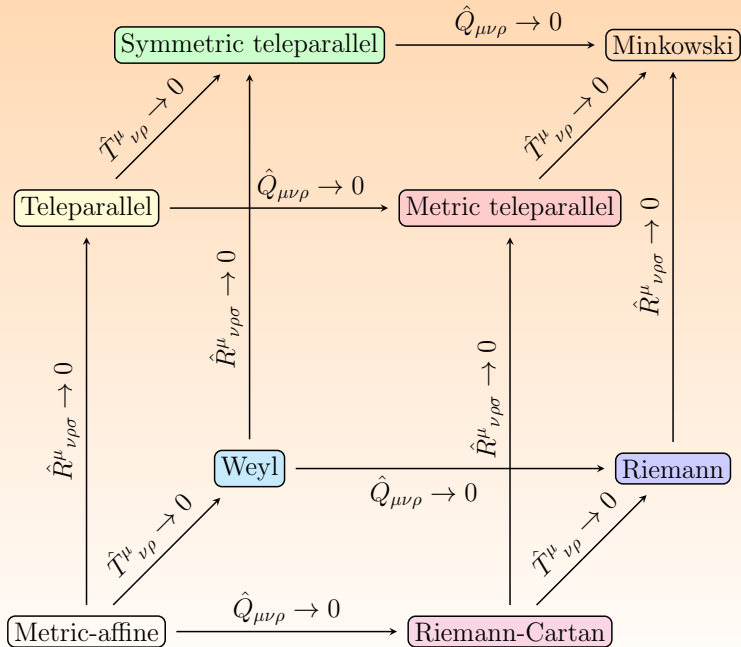
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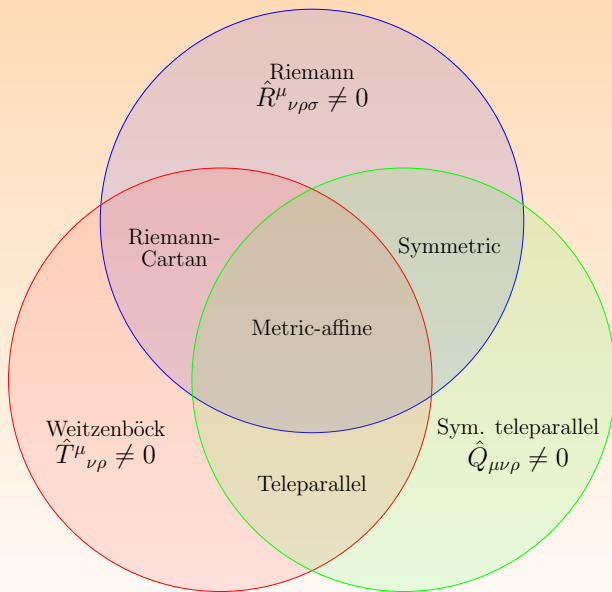
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- **Symmetric Teleparallel geometry** ( $\tilde{R}_{\alpha\mu\nu\beta} = 0, \tilde{T}^\alpha{}_{\mu\nu} = 0$ ): Both torsion tensor and curvature are zero and the gravitational interactions are only mediated through non-metricity.





**Figure:** Classification of metric-affine geometries - Cube





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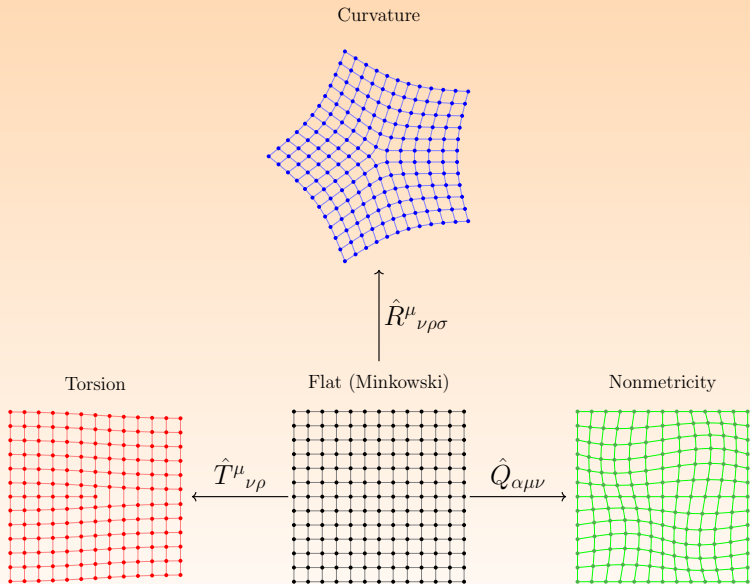
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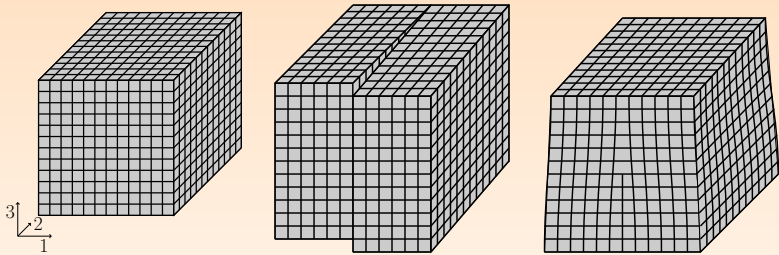
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- **Nonmetricity:** can be understood as crystalline structure with point defects (vacancies/interstitials)





**Figure:** Crystalline structure and its analogy with curvature, torsion and non-metricity





**Figure:** Crystal dislocations are shown against a regular crystal with no dislocation (left), and where *screw* (middle) and *edge* (right) dislocations are represented.



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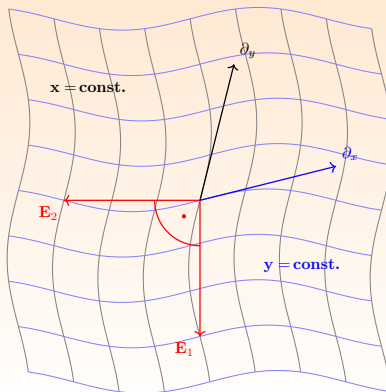
where  $\eta_{ab} = \text{diag}(1, -1, -1, -1)$  denotes the Minkowski metric.



# Tetrads fields

The vectors  $(\mathbf{E}_A)$  form an orthonormal basis of the tangent space, i.e.,

$$g(\mathbf{E}_a, \mathbf{E}_b) = g_{\mu\nu} E_a^\mu E_b^\nu = \eta_{ab}. \quad (2)$$





# Spin connection and tetrads

- The frame coefficients  $E_a{}^\mu$  are also required in order to calculate the coefficients  $\tilde{\Gamma}^\mu{}_{\nu\rho}$  of the affine connection from the spin connection  $\tilde{\omega}^a{}_{b\mu}$  via

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$$\tilde{\Gamma}^\rho_{\mu\nu} = E_a^\rho \left( \partial_\nu e^a_\mu + \tilde{\omega}^a_{b\nu} e^b_\mu \right), \quad (3)$$

- This is the unique affine connection satisfying the so-called “tetrad postulate”

$$\partial_\mu e^a_\nu + \tilde{\omega}^a_{b\mu} e^b_\nu - \tilde{\Gamma}^\rho_{\nu\mu} e^a_\rho = 0. \quad (4)$$



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- One advantage of the formulation in terms of a tetrad and spin connection, is the fact that the curvature, torsion and non-metricity become properties of the spin connection only, and are independent of the choice of the tetrad.
- Then, we can define the curvature, torsion and nonmetricity as:

$$\tilde{R}^a{}_{b\mu\nu} := \partial_\mu \tilde{\omega}^a{}_{b\nu} - \partial_\nu \tilde{\omega}^a{}_{b\mu} + \tilde{\omega}^a{}_{c\mu} \tilde{\omega}^c{}_{b\nu} - \tilde{\omega}^a{}_{c\nu} \tilde{\omega}^c{}_{b\mu}, \quad (5)$$

$$\tilde{T}^a{}_{\mu\nu} := \partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu + \tilde{\omega}^a{}_{b\mu} e^b{}_\nu - \tilde{\omega}^a{}_{b\nu} e^b{}_\mu, \quad (6)$$

$$\tilde{Q}_{\mu ab} := -\eta_{ac} \tilde{\omega}^c{}_{b\mu} - \eta_{cb} \tilde{\omega}^c{}_{a\mu}. \quad (7)$$



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# Trinity of gravity

As mentioned before, we can split the connection as

$$\tilde{\Gamma}^\rho{}_{\mu\nu} := \Gamma^\rho{}_{\mu\nu} + \tilde{K}^\rho{}_{\mu\nu} + \tilde{L}^\rho{}_{\mu\nu} := \Gamma^\rho{}_{\mu\nu} + \tilde{D}^\rho{}_{\mu\nu}, \quad (8)$$

where

$$\Gamma^\mu{}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} (\partial_\nu g_{\sigma\rho} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho}), \quad \text{Levi Civita connection}$$

$$\tilde{K}^\mu{}_{\nu\rho} = \frac{1}{2} \left( \tilde{T}_\nu{}^\mu{}_\rho + \tilde{T}_\rho{}^\mu{}_\nu - \tilde{T}^\mu{}_{\nu\rho} \right), \quad \text{Contortion tensor}$$

$$\tilde{L}^\mu{}_{\nu\rho} = \frac{1}{2} \left( \tilde{Q}^\mu{}_{\nu\rho} - \tilde{Q}_\nu{}^\mu{}_\rho - \tilde{Q}_\rho{}^\mu{}_\nu \right), \quad \text{Disformation tensor}.$$



## Trinity of gravity - curvature tensor

- The curvature becomes

$$\tilde{R}^{\mu}{}_{\nu\rho\sigma} = R^{\mu}{}_{\nu\rho\sigma} + \overset{\circ}{\nabla}_{\rho}\tilde{D}^{\mu}{}_{\nu\sigma} - \overset{\circ}{\nabla}_{\sigma}\tilde{D}^{\mu}{}_{\nu\rho} + \tilde{D}^{\mu}{}_{\tau\rho}\tilde{D}^{\tau}{}_{\nu\sigma} - \tilde{D}^{\mu}{}_{\tau\sigma}\tilde{D}^{\tau}{}_{\nu\rho}.$$



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- Now, by contracting the curvature tensor to obtain the Ricci scalar  $\tilde{R} = g^{\mu\nu}\tilde{R}^{\rho}{}_{\mu\rho\nu}$  we find



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## Ricci scalar decomposition

$$\tilde{R} = R + \left( T + 2\nabla_{\mu}(\sqrt{-g}T^{\rho}{}_{\rho}{}^{\mu}{}_{\mu}) \right) + \left( Q + \nabla_{\mu}Q^{\mu\nu}{}_{\nu} - \nabla_{\nu}Q^{\mu}{}_{\mu}{}^{\nu}{}_{\nu} \right) + C$$



# Trinity of gravity - curvature tensor

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$$\tilde{R}^\mu{}_{\nu\rho\sigma} = R^\mu{}_{\nu\rho\sigma} + \overset{\circ}{\nabla}_\rho \tilde{D}^\mu{}_{\nu\sigma} - \overset{\circ}{\nabla}_\sigma \tilde{D}^\mu{}_{\nu\rho} + \tilde{D}^\mu{}_{\tau\rho} \tilde{D}^\tau{}_{\nu\sigma} - \tilde{D}^\mu{}_{\tau\sigma} \tilde{D}^\tau{}_{\nu\rho}.$$

- Now, by contracting the curvature tensor to obtain the Ricci scalar  $\tilde{R} = g^{\mu\nu} \tilde{R}^\rho{}_{\mu\rho\nu}$  we find

## Ricci scalar decomposition

$$\tilde{R} = R + \left( T + 2\nabla_\mu (\sqrt{-g} T^\rho{}_\rho{}^\mu) \right) + \left( Q + \nabla_\mu Q^{\mu\nu}{}_\nu - \nabla_\nu Q_\mu{}^{\mu\nu} \right) + C$$



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with

$$T := T^{\rho\lambda\kappa} T_{\rho\lambda\kappa} + 2T^{\rho\lambda\kappa} T_{\kappa\rho\lambda} - 4T_\rho{}^\kappa{}_\kappa T^{\rho\lambda}{}_\lambda, \quad \text{Torsion scalar},$$

$$Q := -\frac{1}{4} Q_{\alpha\beta\gamma} Q^{\alpha\beta\gamma} + \frac{1}{2} Q_{\alpha\beta\gamma} Q^{\beta\alpha\gamma} + \frac{1}{4} Q_\alpha Q^\alpha - \frac{1}{2} Q_\alpha \bar{Q}^\alpha, \quad \text{Nonmetricity scalar}$$

$$C := 2(Q_{\kappa\rho\lambda} T^{\lambda\kappa\rho} + Q_\rho{}^\sigma T^{\rho\kappa}{}_\kappa - Q^\sigma{}_{\sigma\rho} T^{\rho\kappa}{}_\kappa).$$



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where  $\kappa^2 = 8\pi G$  and  $L_{\text{m}}$  is any matter Lagrangian.

- The Einstein's field equations are obtained by taking

variations w/r to the metric:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} = \kappa^2 T_{\mu\nu}.$$



## Trinity of gravity - Teleparallel equivalent of GR

- Teleparallel equivalent of GR (TEGR) assumes zero curvature and zero nonmetricity so that



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- Since  $R$  differs by  $T$  by a boundary term  $B_T$ , **the equations of TEGR are equivalent to the Einstein's field equations.**



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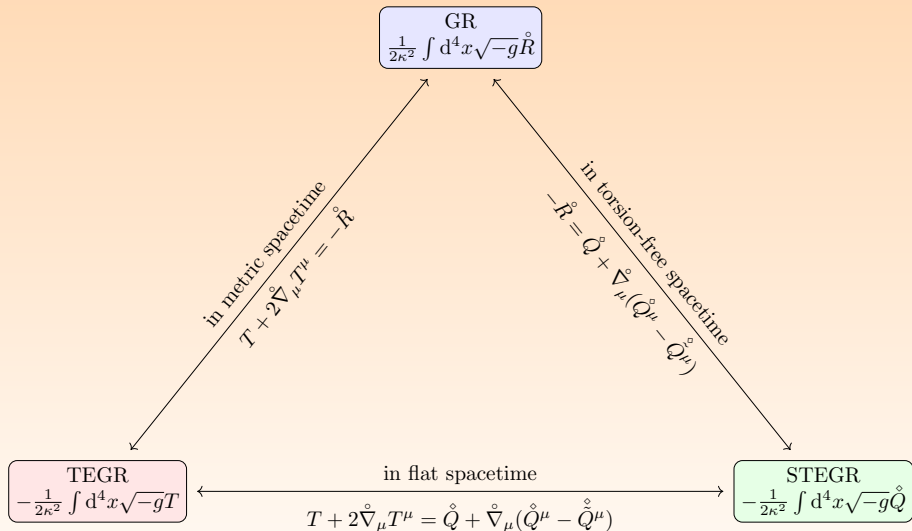
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**Figure:** Geometrical trinity of gravity (S. Bahamonde et.al., “Teleparallel Gravity: From Theory to Cosmology,” [arXiv:2106.13793 [gr-qc]].)



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- Not only an energy-momentum tensor of matter arises, but also a nontrivial spin density tensor which operates as source of torsion  $\implies$  an extended correspondence between the geometry of the space-time and the properties of matter.



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$$A_\mu = e^a{}_\mu P_a + \omega^a{}_{b\mu} L_a{}^b, \quad (9)$$

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- Generators of the group  $A(4, \mathcal{R})$ :

$$[P_a, P_b] = 0, \quad (11)$$

$$[L_a{}^b, P_c] = i \delta^b{}_c P_a, \quad (12)$$

$$[L_a{}^b, L_c{}^d] = i \left( \delta^b{}_c L_a{}^d - \delta_a{}^d L_c{}^b \right). \quad (13)$$



# Gauge formalism of metric-affine geometry

- it is possible to obtain the following gauge curvatures from the anholonomic metric, coframe and connection:

$$G_{ab\mu} = \partial_{\mu} g_{ab} - g_{ac} \omega^c{}_{b\mu} - g_{bc} \omega^c{}_{a\mu} , \quad (14)$$

$$F^a{}_{\mu\nu} = \partial_{\mu} e^a{}_{\nu} - \partial_{\nu} e^a{}_{\mu} + \omega^a{}_{b\mu} e^b{}_{\nu} - \omega^a{}_{b\nu} e^b{}_{\mu} , \quad (15)$$

$$F^a{}_{b\mu\nu} = \partial_{\mu} \omega^a{}_{b\nu} - \partial_{\nu} \omega^a{}_{b\mu} + \omega^a{}_{c\mu} \omega^c{}_{b\nu} - \omega^a{}_{c\nu} \omega^c{}_{b\mu} . \quad (16)$$



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- Correspondence with the curvature, torsion and nonmetricity tensors:

$$G_{ab\mu} = g_{ac} g_{bd} e^{c\lambda} e^{d\rho} Q_{\mu\lambda\rho} , \quad (17)$$

$$F^a{}_{\mu\nu} = e^a{}_\lambda T^\lambda{}_{\nu\mu} , \quad (18)$$

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# Dynamics of metric-affine geometry

- Gravitational action with dynamical torsion and nonmetricity:

$$S = \int d^4x \sqrt{-g} \left[ \mathcal{L}_m - \frac{1}{16\pi} \mathcal{L}_g(\tilde{\mathcal{R}}, \mathcal{T}, \mathcal{Q}) \right]. \quad (20)$$



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- Correspondence between geometry and matter:

$$\frac{\delta S_g}{\delta e^a{}_\nu} = 16\pi \theta_a{}^\nu, \quad (21)$$

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Here  $\theta_a{}^\nu$  is the energy-momentum tensor (canonical) and  $\Delta_a{}^{b\nu}$  is the hypermomentum density tensor.



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- $GL(4, R)$  group allows the definition of a large number of scalar invariants depending on the aforementioned tensors.



# Dynamics of metric-affine geometry

- General quadratic gravitational action with dynamical torsion and nonmetricity:

$$\begin{aligned}
 S = \int d^4x \sqrt{-g} \Big\{ & \mathcal{L}_m + \frac{1}{16\pi} \Big[ -\tilde{R} + a_1 \tilde{R}^2 + a_2 \tilde{R}_{\lambda\rho\mu\nu} \tilde{R}^{\lambda\rho\mu\nu} + a_3 \tilde{R}_{\lambda\rho\mu\nu} \tilde{R}^{\rho\lambda\mu\nu} \\
 & + a_4 \tilde{R}_{\lambda\rho\mu\nu} \tilde{R}^{\mu\nu\lambda\rho} + a_5 \tilde{R}_{\lambda\rho\mu\nu} \tilde{R}^{\lambda\mu\rho\nu} + a_6 \tilde{R}_{\lambda\rho\mu\nu} \tilde{R}^{\mu\lambda\rho\nu} + a_7 \tilde{R}_{\rho\lambda\mu\nu} \tilde{R}^{\mu\lambda\rho\nu} \\
 & + a_8 \tilde{R}_{\mu\nu} \tilde{R}^{\mu\nu} + a_9 \tilde{R}_{\mu\nu} \tilde{R}^{\nu\mu} + a_{10} \hat{R}_{\mu\nu} \hat{R}^{\mu\nu} + a_{11} \hat{R}_{\mu\nu} \hat{R}^{\nu\mu} + a_{12} \tilde{R}_{\mu\nu} \hat{R}^{\mu\nu} \\
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 & + b_1 T_{\lambda\mu\nu} T^{\lambda\mu\nu} + b_2 T_{\lambda\mu\nu} T^{\mu\lambda\nu} + b_3 T^\lambda{}_{\lambda\nu} T^\mu{}_{\mu}{}^\nu + c_1 T_{\lambda\mu\nu} Q^{\mu\lambda\nu} \\
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 & + d_3 Q^\lambda{}_{\lambda\nu} Q^\mu{}_{\mu}{}^\nu + d_4 Q_\nu{}^\lambda{}_{\lambda} Q^{\nu\mu}{}_{\mu} + d_5 Q^\lambda{}_{\lambda\nu} Q^{\nu\mu}{}_{\mu} \Big] \Big\} . \tag{23}
 \end{aligned}$$



# MAG models with dynamical torsion and nonmetricity

- In order to have a theory such that when  $T = Q = 0$  one recovers GR, one can relate the constants.

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<sup>2</sup>S. Bahamonde and J. G. Valcarcel, JCAP **09**, 057 (2020).

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- In order to have a theory such that when  $T = Q = 0$  one recovers GR, one can relate the constants.
- Quadratic gravitational action with dynamical torsion and nonmetricity in Weyl-Cartan geometry ( $Q_{\lambda\mu\nu} = g_{\mu\nu}W_\lambda$ )

$$S = \int d^4x \sqrt{-g} \left\{ \mathcal{L}_m + \frac{1}{64\pi} \left[ -4R - 6d_1 \tilde{R}_{\lambda[\rho\mu\nu]} \tilde{R}^{\lambda[\rho\mu\nu]} \right. \right. \\ \left. - 9d_1 \tilde{R}_{\lambda[\rho\mu\nu]} \tilde{R}^{\mu[\lambda\nu\rho]} + 8d_1 \tilde{R}_{[\mu\nu]} \tilde{R}^{[\mu\nu]} + \frac{1}{8} (32e_1 + 8e_2 + 17d_1) \tilde{R}^\lambda{}_{\lambda\mu\nu} \tilde{R}^\rho{}_\rho{}^{\mu\nu} \right. \\ \left. \left. - 7d_1 \tilde{R}_{[\mu\nu]} \tilde{R}^\lambda{}_{\lambda}{}^{\mu\nu} + 3(1 - 2a_2) T_{[\lambda\mu\nu]} T^{[\lambda\mu\nu]} \right] \right\}.$$

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- Absence of a general Birkhoff's theorem in MAG: new spherically and axially symmetric vacuum solutions with independent dynamical torsion and nonmetricity fields<sup>2,3</sup>

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# Spherical symmetry

- Metric, torsion and nonmetricity in spherically symmetric space-times ( $\#2 + \#8 + \#2 = \#12$ ):

$$\mathcal{L}_\xi g_{\mu\nu} = \mathcal{L}_\xi T^\lambda{}_{\mu\nu} = \mathcal{L}_\xi W_\mu = 0 \implies \mathcal{L}_\xi \tilde{R}_{\lambda\rho\mu\nu} = 0$$



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- By solving these equations we find that torsion and nonmetricity behave as

$$\begin{aligned} T^t{}_{tr} &= a(r), & T^r{}_{tr} &= b(r), & T^{\theta_k}{}_{t\theta_k} &= f(r), & T^{\theta_k}{}_{r\theta_k} &= g(r) \\ T^{\theta_k}{}_{t\theta_l} &= e^{a\theta_k} e^b{}_{\theta_l} \epsilon_{ab} d(r), & T^{\theta_k}{}_{r\theta_l} &= e^{a\theta_k} e^b{}_{\theta_l} \epsilon_{ab} h(r), \\ T^t{}_{\theta_k\theta_l} &= \epsilon_{kl} k(r) \sin \theta_1, & T^r{}_{\theta_k\theta_l} &= \epsilon_{kl} l(r) \sin \theta_1, \\ W_\lambda &= (w_1(r), w_2(r), 0, 0), \end{aligned}$$

whereas the metric is in the standard spherically symmetric form:

$$ds^2 = \Psi_1(r) dt^2 - \frac{dr^2}{\Psi_2(r)} - r^2 (d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2).$$

Here,  $\epsilon_{kl}$  is the Levi-Civita symbol in two dimensions.



# Spherical symmetry - Solving the field equations

The field eqs are very involved. To solve them we use the following strategy:

- 1 **Imposing regularity:** In general, the solutions can have a singular behaviour. To ensure regularity, one can analyse the torsion/nonmetricity tensors referred to the rotated basis  $\vartheta^a = \Lambda^a_b e^b$ .



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One can write the gauge curvature  $\mathcal{F}^a_{bc} = \vartheta^a_\lambda \vartheta_b^\mu \vartheta_c^\nu T^\lambda_{\nu\mu}$  related to the torsion/nonmetricity tensor in this orthogonal coframe.

Regularity restricts the initial arbitrariness of the torsion components and the Weyl vector by imposing the relations

$$\begin{aligned} b(r) &= a(r) \sqrt{\Psi_1(r)\Psi_2(r)}, & f(r) &= -g(r) \sqrt{\Psi_1(r)\Psi_2(r)}, \\ d(r) &= -h(r) \sqrt{\Psi_1(r)\Psi_2(r)}, & l(r) &= k(r) \sqrt{\Psi_1(r)\Psi_2(r)}, \\ w_1(r) &= -w_2(r) \sqrt{\Psi_1(r)\Psi_2(r)}. \end{aligned}$$



# Spherical symmetry - Solving the field equations

- 2 **Solve the weak field limit:** The weak field limit of the field equations become

$$\begin{aligned}\nabla_\rho \nabla_\lambda T^{\lambda\rho}{}_\mu + \nabla_\rho \nabla^\rho T^\lambda{}_{\mu\lambda} - \nabla_\rho \nabla_\mu T^{\lambda\rho}{}_\lambda &= 0, \\ \nabla_\mu \tilde{R}^\lambda{}_\lambda{}^{\mu\nu} &= 0.\end{aligned}$$



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These equations can be solved, yielding

$$\begin{aligned}w_1(r) &= -\kappa_d \int \sqrt{\frac{\Psi_1(r)}{\Psi_2(r)}} \frac{dr}{r^2}, \\ b(r) &= r f'(r) + f(r) + \frac{\kappa_d}{2r} \sqrt{\frac{\Psi_1(r)}{\Psi_2(r)}},\end{aligned}$$

where  $\kappa_d$  is an integration constant which represents the dilaton charge.



# Spherical symmetry - Solving the field equations

- 1 The final solution for the metric behaves as  
Reissner-Nordström

$$g_{tt} = -1/g_{rr} \equiv \Psi(r) = 1 - \frac{2m}{r} + \frac{d_1 \kappa_s^2 - 4e_1 \kappa_{d,e}^2}{r^2}. \quad (24)$$



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- 2 Nonmetricity sector:

$$W_\mu = \frac{\kappa_{d,e}}{r} (1, -1/\Psi(r), 0, 0) . \quad (25)$$



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- 3 Torsion sector:

$$\bar{S}^a = -\frac{\kappa_s}{r} (1, 1, 0, 0), \quad (26)$$

$$\bar{\mathcal{T}}_2^{abc} = \frac{\kappa_s}{3r} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}. \quad (27)$$



# Dilation and spin charges

What do  $\kappa_s$  (dilation charge) and  $\kappa_{d,e}$  (spin charge) physically represent?

## Point 1 - Hypermomentum density

In the geometric scheme of MAG, not only an energy-momentum tensor of matter arises as source of curvature, but also a hypermomentum density tensor which operates as source of torsion and nonmetricity.



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## Point 2 - Dilation and spin charges

In Weyl-Cartan geometry, hypermomentum density tensor splits into spin and dilation currents, which carry their own charges and provide a RN solution.



# Dilation and spin charges

When these charges might be important?

Significant effects are contemplated only around **extreme gravitational systems**, such as **neutron stars** with intense magnetic fields and sufficiently oriented elementary spins or **black holes** endowed with spin and dilation charges.



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# Nature of the black hole solution

- The nature of the horizons depends on the difference  $d_1\kappa_s^2 - 4e_1\kappa_d^2$ . Thus, a positive difference of this quantity would present two horizons determined from the roots

$$r_{\pm} = M \pm \Delta_1, \quad \Delta_1^2 = M^2 - (d_1\kappa_s^2 - 4e_1\kappa_d^2),$$

with  $0 < (d_1\kappa_s^2 - 4e_1\kappa_d^2) < M^2$ .



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- The different signs of the kinetic terms related to the dynamical part of torsion and to the Weyl vector allows the case  $d_1\kappa_s^2 - 4e_1\kappa_d^2 < 0$ .
- The balance between  $\kappa_d$  and  $\kappa_s$  **is not restricted** to any special constraint and therefore any of these situations may occur in the presence of torsion and nonmetricity.



# Particle motion in MAG

- The equations of motion of test bodies with microstructure coupled to the torsion and nonmetricity tensors become<sup>4</sup>

$$\dot{p}^{\mu} + \Gamma^{\mu}{}_{\lambda\rho} p^{\lambda} u^{\rho} + N_{[\lambda\rho]}{}^{\mu} p^{\rho} u^{\lambda} + \tilde{R}_{\lambda\rho\sigma}{}^{\mu} \Delta^{\rho\lambda} u^{\sigma} = 0.$$

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<sup>4</sup>D. Puetzfeld and Y. N. Obukhov, Phys. Rev. D **76** (2007), 084025.



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- This eq. reduces to the standard geodesic one ( $\dot{p}^\mu + \Gamma^\mu{}_{\lambda\rho} p^\lambda u^\rho = 0$ ) when the hypermomentum of the test body vanishes and also when the particle are bosons.

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- Using the standard geodesic approach, we find

$$\frac{1}{2}\dot{r}^2 + V(r) = 0, \quad V(r) = -\frac{1}{2}c^2 E^2 + \frac{1}{2}\Psi(r) \left( \frac{J^2}{r^2} + \sigma c^2 \right),$$

where  $E$  and  $J$  are the conserved charges and  $\sigma = 0 (\sigma = 1)$  represents massless (massive) particles.

<sup>4</sup>D. Puetzfeld and Y. N. Obukhov, Phys. Rev. D **76** (2007), 084025.



# Photon sphere and perihelion shift

- **Photon sphere:** Region of space where gravity is so strong that photons are forced to travel in orbits. They are found by setting  $V'(r) = V(r) = \sigma = 0$ , giving us:

$$r_1 = \frac{G}{2c^2} (3M + \Delta_2) , \quad J_{1,\pm} = \pm \frac{GE(\Delta_2 + 3M)^2}{\sqrt{2}c\sqrt{\Delta_2^2 + 3M^2 + 4\Delta_2 M}} ,$$

$$r_2 = \frac{G}{2c^2} (3M - \Delta_2) , \quad J_{2,\pm} = \pm \frac{GE(\Delta_2 - 3M)^2}{\sqrt{2}c\sqrt{\Delta_2^2 + 3M^2 - 4\Delta_2 M}} ,$$

where we have defined

$$\Delta_2^2 := M^2 + 8\Delta_1^2 = 9M^2 - \frac{8}{G} (d_1\kappa_s^2 - 4e_1\kappa_d^2) .$$



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- The first pair  $(r_1, J_{1,\pm})$  describes a unique photon sphere that lies outside the event horizons, with the corrections related to  $\kappa_s, \kappa_d$  affecting its location with respect to the Schwarzschild solution of GR.



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$$\Delta\phi = 2\pi \left[ \frac{3GM}{c^2 r_c} + \frac{27G^2 M^2}{2c^4 r_c^2} + \frac{135G^3 M^3}{2c^6 r_c^3} + \frac{2835G^4 M^4}{8c^8 r_c^4} - d_1 \kappa_s^2 \left( \frac{1}{2c^2 M r_c} + \frac{6G}{c^4 r_c^2} \right) + e_1 \kappa_d^2 \left( \frac{2}{M c^2 r_c} + \frac{24G}{c^4 r_c^2} \right) \right].$$



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- One could expect that these contributions coming from metric-affine geometry will be only sourced in a strong gravitational regime, e.g., Sgr A\* and S2 stars.

- Since  $\Delta\phi_{S_2}^{(GR)} \approx 48.550 [''/\text{year}]$  and

$\Delta\phi_{S_2}^{(obs)} = 48.506 f_{SP} [''/\text{year}]^5$  we find the constrain

$$4e_1 \kappa_d^2 - 5.711 \cdot 10^{63} [J \cdot m] \leq d_1 \kappa_s^2 \leq 4e_1 \kappa_d^2 + 2.894 \cdot 10^{63} [J \cdot m].$$

<sup>5</sup> B. Abuter *et al.* [GRAVITY], *Astron. Astrophys.* **636** (2020), L5



# Gravitational redshift

- Another gravitational effect that can be used to constrain the new effects arising from our model is the gravitational redshift, that for our solution we get

$$z = \frac{GM}{c^2 R} + \frac{3G^2 M^2}{2c^4 R^2} + \frac{5G^3 M^3}{2c^6 R^3} + \frac{35G^4 M^4}{8c^8 R^4} + \frac{G(4e_1 \kappa_d^2 - d_1 \kappa_s^2)}{2c^4 R^2}.$$



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- We can now consider astrophysical compact objects, such as degenerate stars composed by fermionic matter with a spin alignment induced by torsion.
- Current measurements for the masses and gravitational redshifts of isolated neutron stars do not provide independent quantities.
- We focused on the Sirius B white dwarf which gives us

$$4e_1 \kappa_d^2 - 2.931 \cdot 10^{43} [J \cdot m] \leq d_1 \kappa_s^2 \leq 4e_1 \kappa_d^2 + 1.016 \cdot 10^{43} [J \cdot m].$$



# Observational constraints

- Let us now consider the case where the effect of torsion dominates over the contribution of nonmetricity.

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<sup>6</sup>S. Bahamonde and J. Gigante Valcarcel, Eur. Phys. J. C **81** (2021) no.6, 495.



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- Let us now consider the case where the effect of torsion dominates over the contribution of nonmetricity.
- Indeed, due to the presence of a magnetic field in white dwarfs, it is expected that Sirius B can have sufficiently oriented elementary spins in comparison with an effective dilation charge, therefore,  $\kappa_{s,\text{SiriusB}} \gg \kappa_{d,\text{SiriusB}}$ .

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- Perihelion shift+ Gravitational redshift:** Assuming the same approximation in Sgr A\* and considering the universality of the coupling constant  $d_1$ , we find<sup>6</sup>

$$1.396 \cdot 10^{10} \leq \frac{\kappa_{s,\text{SgrA*}}}{\kappa_{s,\text{SiriusB}}} \leq 1.688 \cdot 10^{10}.$$

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- To the best of our knowledge, this bound provides the first observational comparison between the spin charges of a supermassive black hole and a degenerate star.

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## Extension to axisymmetric space-times

- Metric, torsion and nonmetricity tensors in symmetric space-times:

$$\mathcal{L}_\xi g_{\mu\nu} = \mathcal{L}_\xi T^\lambda{}_{\mu\nu} = \mathcal{L}_\xi Q^\lambda{}_{\mu\nu} = 0 \implies \mathcal{L}_\xi \tilde{R}^\lambda{}_{\rho\mu\nu} = 0. \quad (28)$$



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- Stationary and axisymmetric space-times:

$$\#10 \rightarrow \#4 \left\{ \begin{array}{l} ds^2 = \Psi_1(r, \vartheta) dt^2 - \frac{dr^2}{\Psi_2(r, \vartheta)} \\ - r^2 \Psi_3(r, \vartheta) \left[ d\vartheta^2 + \sin^2 \vartheta (d\varphi - \Psi_4(r, \vartheta) dt)^2 \right] \end{array} \right. ;$$

$$\#24 \left\{ T^\lambda{}_{\mu\nu} = T^\lambda{}_{\mu\nu}(r, \vartheta) \right. \quad (29)$$

$$\#4 \left\{ W_\mu = (W_t(r, \vartheta), W_r(r, \vartheta), W_\vartheta(r, \vartheta), W_\varphi(r, \vartheta)) \right. \quad (30)$$



# Axisymmetric space-times - Kerr-Newmann de-Sitter

## ● Rotating Kerr-Newman metric structure<sup>7</sup>:

$$\begin{aligned}
 ds^2 = & \Psi(r, \vartheta) dt^2 - \frac{r^2 + a^2 \cos^2 \vartheta}{(r^2 + a^2 \cos^2 \vartheta) \Psi(r, \vartheta) + a^2 \sin^2 \vartheta} dr^2 \\
 & - (r^2 + a^2 \cos^2 \vartheta) d\vartheta^2 + 2a (1 - \Psi(r, \vartheta)) \sin^2 \vartheta dt d\varphi \\
 & - \sin^2 \vartheta [r^2 + a^2 + a^2 (1 - \Psi(r, \vartheta)) \sin^2 \vartheta] d\varphi^2, \quad (31)
 \end{aligned}$$

$$\Psi(r, \vartheta) = 1 - \frac{[2mr + 4e_1(\kappa_{d,e}^2 + \kappa_{d,m}^2) - d_1 \kappa_s^2]}{r^2 + a^2 \cos^2 \vartheta}. \quad (32)$$

<sup>7</sup> S. Bahamonde and J. G. Valcarlos, JCAP **01** (2022) no.01, 011.



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$$\begin{aligned}
 ds^2 = & \Psi(r, \vartheta) dt^2 - \frac{r^2 + a^2 \cos^2 \vartheta}{(r^2 + a^2 \cos^2 \vartheta) \Psi(r, \vartheta) + a^2 \sin^2 \vartheta} dr^2 \\
 & - (r^2 + a^2 \cos^2 \vartheta) d\vartheta^2 + 2a (1 - \Psi(r, \vartheta)) \sin^2 \vartheta dt d\varphi \\
 & - \sin^2 \vartheta [r^2 + a^2 + a^2 (1 - \Psi(r, \vartheta)) \sin^2 \vartheta] d\varphi^2,
 \end{aligned} \tag{31}$$

$$\Psi(r, \vartheta) = 1 - \frac{[2mr + 4e_1(\kappa_{d,e}^2 + \kappa_{d,m}^2) - d_1\kappa_s^2]}{r^2 + a^2 \cos^2 \vartheta}. \tag{32}$$

## • Field strength tensors:

$$\begin{aligned}
 \bar{R}_{[\mu\nu]} &= \frac{1}{12} \varepsilon^\lambda{}_{\sigma\mu\nu} \nabla_\lambda \bar{S}^\sigma + \frac{1}{2} \nabla_\lambda \bar{t}^\lambda{}_{\mu\nu}; \quad \tilde{R}^\lambda{}_{\lambda\mu\nu} = 4 \nabla_{[\nu} W_{\mu]}; \\
 \bar{R}^\lambda{}_{[\mu\nu\rho]} &= \frac{1}{6} \varepsilon^\lambda{}_{\sigma[\rho\nu} \nabla_{\mu]} \bar{S}^\sigma + \nabla_{[\mu} \bar{t}^\lambda{}_{\rho\nu]} + \frac{1}{4} \varepsilon^\lambda{}_{\omega\sigma[\rho} \bar{t}_1^\sigma{}_{\mu\nu]} \bar{S}^\omega \\
 &\quad - \frac{1}{18} \varepsilon_{\sigma\mu\nu\rho} \bar{T}_1^\lambda \bar{S}^\sigma.
 \end{aligned} \tag{33}$$

<sup>7</sup> S. Bahamonde and J. G. Valcarlos, JCAP **01** (2022) no.01, 011.



# Axisymmetric space-times - Kerr-Newmann de-Sitter

## ● Nonmetricity sector:(no approx.)

$$\begin{aligned}
 w_1(r, \vartheta) &= \frac{\kappa_{d,e} r - a \kappa_{d,m} \cos \vartheta}{r^2 + a^2 \cos^2 \vartheta}, \quad w_3(r, \vartheta) = 0, \\
 w_2(r, \vartheta) &= - \frac{\kappa_{d,e} r}{(r^2 + a^2 \cos^2 \vartheta) \Psi(r, \vartheta) + a^2 \sin^2 \vartheta}, \\
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## • Torsion sector (decoupling limit between the spin and the orbital angular momentum $|a\kappa_s| \ll 1$ ):

$$\bar{\mathcal{S}}^a = - \frac{\kappa_s}{r} (1, 1, 0, 0) + \mathcal{O}(a\kappa_s), \quad (35)$$

$$\bar{\mathcal{T}}_2^{abc} = \frac{\kappa_s}{3r} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 \end{pmatrix} + \mathcal{O}(a\kappa_s). \quad (36)$$



# Gravitational spin-orbit interaction

- We found a solution in the decoupling limit  $a\kappa_s \ll 1$ , which ensures that the Maxwell equation and closure conditions are fulfilled by the field strength tensors of torsion

$$\nabla_\lambda \tilde{R}^\lambda{}_{[\rho\mu\nu]} = \nabla_\mu \tilde{R}^{[\mu\nu]} = 0, \quad \nabla_{[\sigma} \tilde{R}_{\lambda[\rho\mu\nu]]} = \nabla_{[\lambda} \tilde{R}_{\mu\nu]} = 0.$$



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- **Gravitational spin-orbit interaction:**

$$\mathcal{H}_{\text{LS}} = \frac{1}{m_e^2 r} \frac{\partial V}{\partial r} \mathbf{L} \cdot \mathbf{S} \approx \frac{d_1}{2r} \frac{\partial g_{tt}}{\partial r} a\kappa_s \cos \vartheta \quad (37)$$



## Extension to axisymmetric space-times - Plebanski-Damianski

- It is well known that the most general axisymmetric system in vacuum that can describe a BH type D in **GR** contains<sup>8</sup>:

<b>Mass</b>	$M$
<b>Angular momentum</b>	$a$
<b>Taub-NUT charge</b>	$l$
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$$ds^2 = \Omega^{-2}(r, \vartheta) \left\{ \Phi_1(r, \vartheta) \left[ dt - (a \sin^2 \vartheta + 2l(\chi - \cos \vartheta)) d\varphi \right]^2 - \frac{dr^2}{\Phi_1(r, \vartheta)} - \frac{d\vartheta^2}{\Phi_2(r, \vartheta)} - \Phi_2(r, \vartheta) \sin^2 \vartheta \left[ a dt - (r^2 + a^2 + l^2 + 2\chi a l) d\varphi \right]^2 \right\}.$$

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- We just found this new form with the cosmological constant<sup>9</sup> with  $\Phi_1(r, \vartheta) = \frac{Q(r)}{\rho^2(r, \vartheta)}$ ,  $\Phi_2(r, \vartheta) = \frac{P(\vartheta)}{\rho^2(r, \vartheta)}$ , and  $\rho^2(r, \vartheta) = r^2 + (a \cos \vartheta + l)^2$ . Here,  $Q(r), \Omega(\vartheta)$  include the PD quantities.

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- We found a solution to **OUR THEORY** in the decoupling limit  $|x_i \kappa_s| \ll 1$  with  $x = (a, l, \alpha)$  with additional torsion and nonmetricity terms



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$$w_1(r, \vartheta) = \frac{\kappa_{d,e} r - \kappa_{d,m}(a \cos \vartheta + l)}{r^2 + (a \cos \vartheta + l)^2}, \quad w_2(r, \vartheta) = -\frac{\kappa_{d,e} r - \kappa_{d,m}(a \gamma + l)}{Q(r)},$$

$$w_3(r, \vartheta) = -\kappa_{d,m} \sqrt{K(\vartheta) - \left( \frac{\cot \vartheta - \gamma \csc \vartheta}{P(\vartheta)} \right)^2},$$

$$w_4(r, \vartheta) = \kappa_{d,m} \left[ \frac{(r^2 + a^2 - l^2) \cos \vartheta + a l \sin^2 \vartheta + 2 \chi l (a \cos \vartheta + l)}{r^2 + (a \cos \vartheta + l)^2} - \gamma \right] \\ - \frac{\kappa_{d,e} r [a \sin^2 \vartheta + 2l(\chi - \cos \vartheta)]}{r^2 + (a \cos \vartheta + l)^2},$$

$$\bar{T}^\vartheta{}_{\varphi t} = -\bar{T}^\varphi{}_{\vartheta t} \sin^2 \vartheta = -\bar{T}^\vartheta{}_{\varphi r} \frac{Q(r)}{\rho^2(r, \vartheta)} = \bar{T}^\varphi{}_{\vartheta r} \frac{Q(r)}{\rho^2(r, \vartheta)} \sin^2 \vartheta = \frac{\kappa_s \sin \vartheta}{r} + \mathcal{O}(x_i \kappa_s).$$

- Similarly as electromagnetism, the torsion behaves as a Coulomb-like quantity depending on a spin charge  $\kappa_s$  and the non-metricity on the dilation charge  $\kappa_d$ .



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- There are three alternative ways of representing gravity as GR and they are indistinguishable at the classical level (GR,TEGR, STEGR).
- The MAG are gauge theories of gravity with the field strength tensors given by the curvature, torsion and nonmetricity.



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- In a 4th paper we found Plebanski-Demianski uniformly accelerated rotating black hole solutions with NUT parameter, electromagnetic charges and a  $\Lambda$ .
- Future: search of a gravitational spin-orbit interaction in MAG beyond the Kerr-Newman space-time (MAG is the main candidate to describe a spin-orbit interaction beyond GR).