Black hole Scalar Hairs in Teleparallel gravity

Sebastián Bahamonde

Postdoctoral Researcher at Tokyo Institute of Technology, Japan

2022 Summer CAS-JSPS Workshop in Cosmology, Gravitation, and Particle Physics, 30/08/2022

Based mainly on JCAP **01** (2022) no.01, JCAP **04** (2022) no.04, 018, arXiv:206.02725 (to appear in JCAP) and our review arXiv:2106.13793

Overview of the Talk

1. Introduction to Teleparallel theories of gravity
   - Basic mathematical ingredients
   - Trinity of gravity
2. Modified Teleparallel theories of gravity
   - General features
3. Black holes in torsional teleparallel gravity
   - Basic considerations
   - Scalarised black holes in scalar-torsion
4. Black holes in symmetric teleparallel gravity (non-metricity)
   - Basic considerations
   - Scalarised black holes in symmetric TG
5. Briefly on Spontaneous scalarization
6. Conclusions and final remarks

Fundamental variables and characteristic tensors

- In the most general metric-affine setting, the fundamental variables are a **metric** $g_{\mu\nu}$ (10 comp.) as well as the coefficients $\hat{\Gamma}^{\rho}{}_{\mu\nu}$ (64 comp.) of an **affine connection**.

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Curvature	$\hat{R}^{\mu}{}_{\nu\rho\sigma} = \partial_{\rho}\hat{\Gamma}^{\mu}{}_{\nu\sigma} - \partial_{\sigma}\hat{\Gamma}^{\mu}{}_{\nu\rho} + \hat{\Gamma}^{\mu}{}_{\tau\rho}\hat{\Gamma}^{\tau}{}_{\nu\sigma} - \hat{\Gamma}^{\mu}{}_{\tau\sigma}\hat{\Gamma}^{\tau}{}_{\nu\rho}$
Torsion	$\hat{T}^{\mu}{}_{\nu\rho} = \hat{\Gamma}^{\mu}{}_{\rho\nu} - \hat{\Gamma}^{\mu}{}_{\nu\rho}$
Nonmetricity	$\hat{Q}_{\mu\nu\rho} = \hat{\nabla}_{\mu}g_{\nu\rho} = \partial_{\mu}g_{\nu\rho} - \hat{\Gamma}^{\sigma}{}_{\nu\mu}g_{\sigma\rho} - \hat{\Gamma}^{\sigma}{}_{\rho\mu}g_{\nu\sigma}$

Tetrads and spin connection

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where η_{ab} is the Minkowski metric.

- Quantities denoted with a circle on top $\circ$ denote that they are defined with respect to the Levi-Civita connection and hats are general affine connection.

Trinity of gravity - curvature tensor

- The curvature becomes

$$\hat{R}^{\mu}{}_{\nu\rho\sigma} = \mathring{R}^{\mu}{}_{\nu\rho\sigma} + \mathring{\nabla}_{\rho}\hat{D}^{\mu}{}_{\nu\sigma} - \mathring{\nabla}_{\sigma}\hat{D}^{\mu}{}_{\nu\rho} + \hat{D}^{\mu}{}_{\tau\rho}\hat{D}^{\tau}{}_{\nu\sigma} - \hat{D}^{\mu}{}_{\tau\sigma}\hat{D}^{\tau}{}_{\nu\rho}\,.$$

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with

$$T := T^{\rho\lambda\kappa}T_{\rho\lambda\kappa} + 2T^{\rho\lambda\kappa}T_{\kappa\rho\lambda} - 4T_{\rho}{}^{\kappa}{}_{\kappa}T^{\rho\lambda}{}_{\lambda}\,, \quad \text{Torsion scalar}\,,$$

$$Q := -\frac{1}{4}Q_{\alpha\beta\gamma}Q^{\alpha\beta\gamma} + \frac{1}{2}Q_{\alpha\beta\gamma}Q^{\beta\alpha\gamma} + \frac{1}{4}Q_{\alpha}Q^{\alpha} - \frac{1}{2}Q_{\alpha}\bar{Q}^{\alpha}\,, \ \text{Nonmetricity scalar}\,,$$

$$C := 2(Q_{\kappa\rho\lambda}T^{\lambda\kappa\rho} + Q_{\rho}{}^{\sigma}{}_{\sigma}T^{\rho\kappa}{}_{\kappa} - Q^{\sigma}{}_{\sigma\rho}T^{\rho\kappa}{}_{\kappa})\,.$$

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where $\kappa^2 = 8\pi G$ and $L_{\rm m}$ is any matter Lagrangian.

- The Einstein's field equations are obtained by taking variations w/r to the metric: $\boxed{\mathring{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathring{R} = \kappa^2 T_{\mu\nu}}$.

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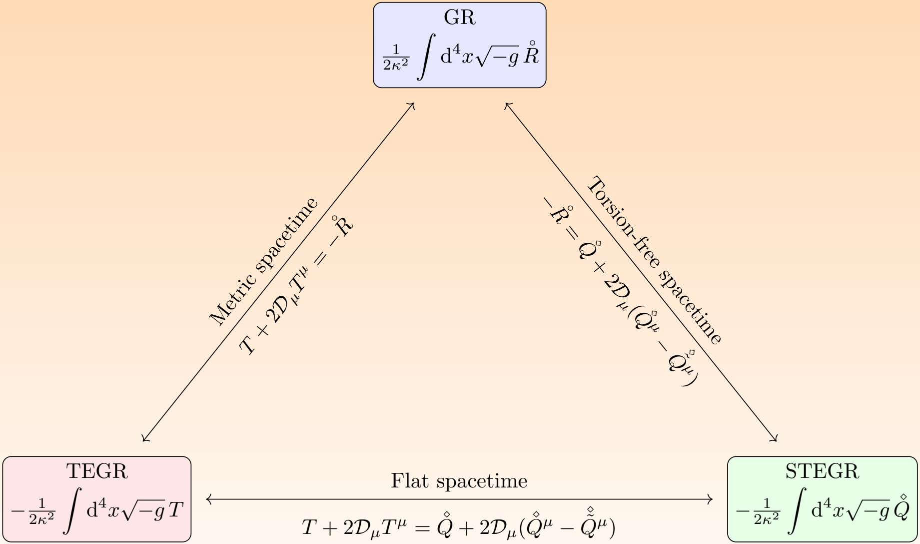

Figure: Geometrical trinity of gravity (S. Bahamonde et.al., “Teleparallel Gravity: From Theory to Cosmology,” [arXiv:2106.13793 [gr-qc]].; J. Beltrán Jiménez, L. Heisenberg and T. S. Koivisto, “The Geometrical Trinity of Gravity,” Universe **5** (2019) no.7, 173.)

Overview of the Talk

Modified teleparallel theories

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Are Teleparallel theories diffeo and local Lorentz invariant?

There is a lot of misconceptions in the literature, but yes both the torsional and nonmetricity versions are fully invariant (diffeo+local Lorentz)

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- It is then **sufficient** to take variations w/r to the tetrad only and all the gravitational information will be encoded in the symmetric and antisymmetric part of those field equations.
- Keep in mind: the metric has 10 d.o.f. and the tetrad $10+6$. Since these theories have $e^a{}_\mu$ as the fundamental variables, **the same metric can have different tetrads**.

Spherical symmetry in torsional TG

- Riemannian geometry (as GR): Killing eq: $\mathcal{L}_{Z_\zeta} g_{\alpha\beta} = 0$ a symmetry and then $\mathcal{L}_{Z_\zeta} \mathring{\Gamma}^{\alpha}{}_{\mu\nu} = 0$. Then, the field equations of the theory satisfy the symmetries.

[1] M. Hohmann, L. Järv, M. Krššák and C. Pfeifer, Phys. Rev. D **100** (2019) no.8, 084002

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- Torsional teleparallel:($\mathcal{L}_{Z_\zeta} g_{\alpha\beta} = 0$ **does not imply** $\mathcal{L}_{Z_\zeta} \Gamma^{\alpha}{}_{\mu\nu} = 0$). Two conditions:

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[1] M. Hohmann, L. Järv, M. Krššák and C. Pfeifer, Phys. Rev. D **100** (2019) no.8, 084002

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By solving these eqs in the Weitzenböck gauge we find[1]

$$e^{A}{}_{\nu} = \begin{pmatrix} C_1 & C_2 & 0 & 0 \\ C_3 \sin\vartheta \cos\varphi & C_4 \sin\vartheta \cos\varphi & C_5 \cos\vartheta \cos\varphi - C_6 \sin\varphi & -\sin\vartheta (C_5 \sin\varphi + C_6 \cos\vartheta \cos\varphi) \\ C_3 \sin\vartheta \sin\varphi & C_4 \sin\vartheta \sin\varphi & C_5 \cos\vartheta \sin\varphi + C_6 \cos\varphi & \sin\vartheta (C_5 \cos\varphi - C_6 \cos\vartheta \sin\varphi) \\ C_3 \cos\vartheta & C_4 \cos\vartheta & -C_5 \sin\vartheta & C_6 \sin^2\vartheta \end{pmatrix}.$$

- By redefining $C_i(t,r)$ we find $ds^2 = A(r)dt^2 - B(r)dr^2 - r^2 d\Omega^2$ and two extra functions only appear in the tetrad. **We can set those two extra with the antisymmetric field equations**.

[1] M. Hohmann, L. Järv, M. Krššák and C. Pfeifer, Phys. Rev. D **100** (2019) no.8, 084002

Scalarised black holes in Riemannian geometry

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- When $\mathcal{F}(\psi) = 1$ and $V \propto -\psi^2$, there is a no-hair theorem.
- One can circumvent the no-hair theorem by having some particular potentials and coupling functions.

Scalar-torsion theories

- One can formulate Teleparallel theories with a scalar field, for example[2]:

$$S = \frac{1}{2\kappa^2}\int_M \left[-\mathcal{A}(\psi)T - \tilde{\mathcal{C}}(\psi)B + 2\mathcal{B}(\psi)X - 2\kappa^2\mathcal{V}(\psi)\right]\sqrt{-g}\,\mathrm{d}^4x\,,$$

where $X = -\frac{1}{2}g^{\mu\nu}\partial_\mu\psi\partial_\nu\psi$.

[2] S. Bahamonde and M. Wright, Phys. Rev. D **92** (2015) no.8, 084034; M. Zubair, S. Bahamonde and M. Jamil, Eur. Phys. J. C **77** (2017) no.7, 472; M. Hohmann and C. Pfeifer, Phys. Rev. D **98** (2018) no.6, 064003.

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- In appropriate limits $f(T)$ or $f(\mathring{R})$ or $f(T,B)$[3] gravity can be obtained from the above action.
- For example $f(T)$ can be found by $\mathcal{B} = \tilde{\mathcal{C}}(\psi) = 0$, $\mathcal{A}(\psi) = f_T$ and $V(\psi) = 1/2(Tf_T - f)$ and $\psi = T$.

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Solving the antisymmetric field equations

- There are two different tetrads which solve the antisymmetric field equation and they have the same metric[4]

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e_{(1)}{}^{A}{}_{\mu} = \begin{pmatrix} \mathcal{A}(r) & 0 & 0 & 0 \\ 0 & \mathcal{B}(r)\sin\vartheta\cos\varphi & \xi r\cos\vartheta\cos\varphi & -r\xi\sin\vartheta\sin\varphi \\ 0 & \mathcal{B}(r)\sin\vartheta\sin\varphi & \xi r\cos\vartheta\sin\varphi & \xi r\sin\vartheta\cos\varphi \\ 0 & \mathcal{B}(r)\cos\vartheta & -r\xi\sin\vartheta & 0 \end{pmatrix}, \quad \xi = \pm 1\,,
$$

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- **The phenomenology of these two tetrads will be different:** For the same metric, two different sets of equations.

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- We found exact solutions for different theories for the two tetrads, some of them with a scalar hair.

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Black hole Solution 1: Born-Infeld $f(T)$

- A quite well-studied theory inspired by Born-Infeld electromagnetism is

$$f(T) = \lambda\left(\sqrt{1+\frac{2T}{\lambda}}-1\right),$$

with λ being the so-called Born-Infeld parameter. It is easy to notice that when $T/\lambda \ll 1$, one obtains $f(T) = T - T^2/(2\lambda) + \mathcal{O}(1/\lambda^2)$.

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- We found an exact black hole solution to this theory(JCAP **01** (2022) no.01, 0370)

$$\begin{aligned}\mathrm{d}s^2 = {} & \frac{a_1^2}{r}\left[\sqrt{\lambda}(a_0\lambda + r) - 2\tan^{-1}\left(\frac{\sqrt{\lambda}r}{2}\right)\right]\mathrm{d}t^2 \\ & - \frac{\lambda^{5/2}r^5}{(4+r^2\lambda)^2}\left[\sqrt{\lambda}(a_0\lambda + r) - 2\tan^{-1}\left(\frac{\sqrt{\lambda}r}{2}\right)\right]^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2\,.\end{aligned}$$

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- This solution behaves similarly to Schwarzschild with one horizon but the horizon depends on M and λ (which acts as an extra hair).

Solution 2: BBMB

- For a theory with a coupling like $\mathcal{A}(\psi)T$ with a kinetic term $\mathcal{B} = \beta$, we found exact solutions for the real and complex tetrad[5].

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- The metric is the same as the Bocharova–Bronnikov–Melnikov–Bekenstein (BBMB) solution found in Riemannian conformal scalar-vacuum theory!
- New solutions that do not appear in the Riemannian case appear in this theories. Also, we also formulated no-hair theorems. See our paper for this.

[5] S. Bahamonde, L. Ducobu and C. Pfeifer, "Scalarized black holes in teleparallel gravity," JCAP **04** (2022) no.04, 018

Overview of the Talk

- The dynamical variables are the **metric** and a **torsionless and flat connection.**

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Modified symmetric teleparallel theories (non-metricity)

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- Keep in mind: the metric has 10 d.o.f. and now we have metric+connection, so that, **there are more d.o.f.**

Spherical symmetry in symmetric TG

- Riemannian geometry (as GR): Killing eq: $\mathcal{L}_{Z_\zeta} g_{\alpha\beta} = 0$ a symmetry and then $\mathcal{L}_{Z_\zeta} \mathring{\Gamma}^{\alpha}{}_{\mu\nu} = 0$. Then, the field equations of the theory satisfy the symmetries.

[6] F. D'Ambrosio, S. D. B. Fell, L. Heisenberg and S. Kuhn, Phys. Rev. D **105** (2022) no.2, 024042

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- By solving these eqs, we find that there are **two different connections** (respecting zero curvature and zero torsion) satisfying spherical symmetry[6]. (set 1 and set 2).

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- By solving these eqs, we find that there are **two different connections** (respecting zero curvature and zero torsion) satisfying spherical symmetry[6]. (set 1 and set 2).
- Similarly as the torsional case, for the **same metric (spherically symmetric)** we will have **two sets of field equations with different phenomenology**.

[6] F. D'Ambrosio, S. D. B. Fell, L. Heisenberg and S. Kuhn, Phys. Rev. D **105** (2022) no.2, 024042

Scalar-tensor theories in symmetric TG

- Similarly as with the torsional case, one can formulate analogues theories in symmetric TG:[7]

$$S=\frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g}\,\Big(\mathcal{A}(\Phi)Q-\mathcal{B}(\Phi)g^{\alpha\beta}\partial_\alpha\Phi\partial_\beta\Phi-2\mathcal{V}(\Phi)\Big)\,,$$

where $Q\equiv-\frac{1}{4}Q_{\lambda\mu\nu}Q^{\lambda\mu\nu}+\frac{1}{2}Q_{\lambda\mu\nu}Q^{\mu\nu\lambda}+\frac{1}{4}Q_{\mu}Q^{\mu}-\frac{1}{2}Q_{\mu}\tilde{Q}^{\mu}$.

[7] L. Järv, M. Rünkla, M. Saal and O. Vilson, Phys. Rev. D **97** (2018) no.12, 124025

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- If $\mathcal{A}(\Phi)=const$, we just have Einstein-Klein-Gordon. Also, the above theory contains $f(Q)$ gravity.

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$$\Phi(r) = \Phi_0 \left(-\frac{M}{rW\left(-\frac{M}{r}\right)}\right)^{1/2} \,, \quad \mathcal{A}(\Phi) = \frac{\beta\Phi^2}{8} \,, \quad \mathcal{V}(\Phi) = 0 \,,$$

where $W(z)$ is the Lambert function defined as $W(z)e^{W(z)} = z$ and M represents the ADM mass of the black hole.

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- No-hair theorems and more new solutions (see our paper).

Overview of the Talk

- One very interesting theory in Riemannian case is

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- This theory has **spontaneous scalarization**: there exist BH solutions which are formed by spontaneous scalarization of the Schwarzaschild BH in the extreme curvature regime.
- In this regime, below certain mass, the Schwarzschild solution becomes unstable and new branch of solutions with nontrivial scalar field bifurcate from the Schwarzschild one.

- The simplest way to see that this process exists is looking at the Klein-Gordon eq: $\mathring{\Box}\phi + \frac{df(\phi)}{d\phi}\mathring{G} = 0$. Since $\mathring{G} \neq 0$ in Schwarzschild, the Gauss-Bonnet invariant sources a scalar-hair.

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- In a theory like $\mathcal{L} = \mathring{R} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \alpha f(\phi)\mathring{R}$, there are not spontaneous scalarization ($\mathring{R} = 0$ in Schwarzschild).

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- In a theory like $\mathcal{L} = \mathring{R} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \alpha f(\phi)\mathring{R}$, there are not spontaneous scalarization ($\mathring{R} = 0$ in Schwarzschild).
- In torsional teleparallel versions: Since T and B are different to zero in Schwarzschild, we expect to have spontaneous scalarization for the simplest theory coupled to $\mathcal{L} = -T - \frac{1}{2}\beta\partial_\mu\phi\partial^\mu\phi - \mathcal{A}(\phi)T + \mathcal{B}(\phi)B$.

- Simple computation show that this could be true: $ds^2 = e^{\delta} A dt^2 - 1/A dr^2 - r^2 d\Omega^2$ with $\delta \ll 1$ and $A = 1 - 2M/r$ and $\delta\psi(t, r, \theta, \psi) = u(r)/r\, e^{-i\omega t}\, Y_{lm}(\theta, \psi)$:

$$\frac{d^2 u}{dr_*^2} + [\omega^2 - U(r)]u = 0\,, \quad \text{with } \mathcal{A}'(0) = \mathcal{B}'(0)\,,$$

where we have introduced tortoise coordinates $dr_* = (1 - 2M/r)^{-1} dr$ and the potential $U(r)$ is

$$U(r) = \left(1 - \frac{2M}{r}\right)\left(\frac{2M}{r^3} + \frac{l(l+1)}{r^2} + \frac{4}{\beta r^2}\left(-\mathcal{A}''(0) + \mathcal{B}''(0)\right)\right).$$

We have an unstable mode if

$$\int_{-\infty}^{+\infty} U(r_*) dr_* = \int_{2M}^{\infty} \frac{U(r)}{1 - \frac{2M}{r}} dr < 0\,,$$

and then $\frac{1}{4M} < -\frac{2(-\mathcal{A}''(0) + \mathcal{B}''(0))}{\beta M}$.

Overview of the Talk

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- We formulated no-hair theorems for scalar-torsion gravity and symmetric (non-metricity) scalar-tensor theories.

Conclusions and final remarks

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- Spontaneous scalarization in Teleparallel gravity?