Metric-Affine theories of gravity and their applications

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Outline

Overview of the Talk

1. Brief introduction to metric-affine geometry
2. Teleparallel theories
   - Trinity of Gravity
   - Modified Theories with torsion and applications (Metric TG)
   - Theories with Nonmetricity and applications (Symmetric TG)
3. Gauge theories of gravity
4. Cosmological perturbations in MAG
5. Algebraic Classification in MAG
6. Stability in Metric Affine Gauge theories
   - Stability issues in quadratic Poincaré gauge theory
   - Cubic extensions
7. Conclusions and future prospects

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- Definition of the curvature tensor:

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- In particular, it measures the change of vector components on parallel transport along an infinitesimal closed curve:

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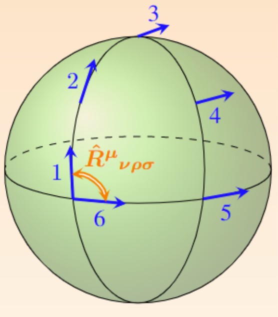

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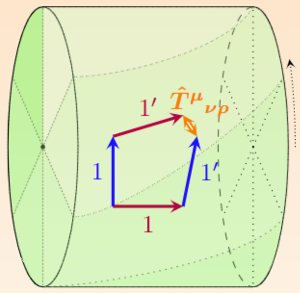

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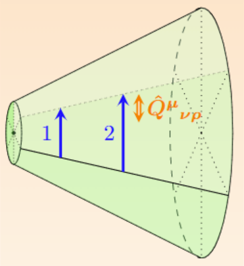

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Post-Riemannian decomposition

- According to the Fundamental Theorem of Riemannian geometry, in the absence of torsion and nonmetricity:

$$\tilde{\Gamma}^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left(\partial_{\mu} g_{\nu\rho} + \partial_{\nu} g_{\mu\rho} - \partial_{\rho} g_{\mu\nu} \right) ,$$

$$\tilde{R}_{\lambda\rho\mu\nu} = R_{\lambda\rho\mu\nu} = \frac{1}{2} \left(\frac{\partial^2 g_{\lambda\nu}}{\partial x^{\rho} \partial x^{\mu}} + \frac{\partial^2 g_{\rho\mu}}{\partial x^{\lambda} \partial x^{\nu}} - \frac{\partial^2 g_{\lambda\mu}}{\partial x^{\rho} \partial x^{\nu}} - \frac{\partial^2 g_{\rho\nu}}{\partial x^{\lambda} \partial x^{\mu}} \right) + g_{\sigma\omega} \left(\Gamma^{\omega}{}_{\rho\mu} \Gamma^{\sigma}{}_{\lambda\nu} - \Gamma^{\omega}{}_{\rho\nu} \Gamma^{\sigma}{}_{\lambda\mu} \right) .$$

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- General decomposition of the affine connection and the curvature tensor:

$$\tilde{\Gamma}^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\mu\nu} + N^{\lambda}{}_{\mu\nu} ,$$

$$N^{\lambda}{}_{\mu\nu} = \frac{1}{2} T^{\lambda}{}_{\mu\nu} - T_{(\mu}{}^{\lambda}{}_{\nu)} + \frac{1}{2} Q^{\lambda}{}_{\mu\nu} - Q_{(\mu}{}^{\lambda}{}_{\nu)} ,$$

$$\tilde{R}^{\lambda}{}_{\rho\mu\nu} = R^{\lambda}{}_{\rho\mu\nu} + 2\nabla_{[\mu|} N^{\lambda}{}_{\rho|\nu]} + 2 N^{\lambda}{}_{\sigma[\mu|} N^{\sigma}{}_{\rho|\nu]} .$$

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\tilde{\nabla}_{[\sigma|} \tilde{R}^{\lambda}{}_{\rho|\mu\nu]} &= T^{\omega}{}_{[\sigma\mu|} \tilde{R}^{\lambda}{}_{\rho\omega|\nu]}\,,\\
\tilde{R}^{(\lambda\rho)}{}_{\mu\nu} &= \tilde{\nabla}_{[\nu} Q_{\mu]}{}^{\lambda\rho} + \frac{1}{2} T^{\sigma}{}_{\mu\nu} Q_{\sigma}{}^{\lambda\rho}\,.
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- Three independent rank-2 tensors defined from the first contractions of the curvature tensor:

$$\tilde{R}_{\mu\nu} = \tilde{R}^{\lambda}{}_{\mu\lambda\nu} \,, \quad \hat{R}_{\mu\nu} = \tilde{R}_{\mu}{}^{\lambda}{}_{\nu\lambda} \,, \quad \tilde{R}^{\lambda}{}_{\lambda\mu\nu} = \nabla_{[\nu} Q_{\mu]\lambda}{}^{\lambda} \,.$$

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- Unique scalar and pseudoscalar curvatures:

$$\tilde{R} = g^{\mu\nu} \tilde{R}_{\mu\nu}\,, \quad *\tilde{R} = \varepsilon^{\lambda\rho\mu\nu} \tilde{R}_{\lambda\rho\mu\nu}\,.$$

Overview of the Talk

Teleparallel theories

- Teleparallel condition:

$$\tilde{R}^{\mu}{}_{\nu\rho\sigma} = R^{\mu}{}_{\nu\rho\sigma} + \nabla_{\rho} N^{\mu}{}_{\nu\sigma} - \nabla_{\sigma} N^{\mu}{}_{\nu\rho} + N^{\mu}{}_{\tau\rho} N^{\tau}{}_{\nu\sigma} - N^{\mu}{}_{\tau\sigma} N^{\tau}{}_{\nu\rho} = 0\,.$$

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Ricci scalar decomposition

$$\tilde{R} = R + \left(T + 2\mathring{\nabla}_{\mu}(\sqrt{-g}T^{\rho}{}_{\rho}{}^{\mu})\right) + \left(Q + \mathring{\nabla}_{\mu}Q^{\mu\nu}{}_{\nu} - \mathring{\nabla}_{\nu}Q_{\mu}{}^{\mu\nu}\right) + C = 0$$

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with

$$T := T^{\rho\lambda\kappa}T_{\rho\lambda\kappa} + 2T^{\rho\lambda\kappa}T_{\kappa\rho\lambda} - 4T_{\rho}{}^{\kappa}{}_{\kappa}T^{\rho\lambda}{}_{\lambda}\,, \quad \text{Torsion scalar}\,,$$

$$Q := -\frac{1}{4}Q_{\alpha\beta\gamma}Q^{\alpha\beta\gamma} + \frac{1}{2}Q_{\alpha\beta\gamma}Q^{\beta\alpha\gamma} + \frac{1}{4}Q_{\alpha}Q^{\alpha} - \frac{1}{2}Q_{\alpha}\bar{Q}^{\alpha}\,, \text{ Nonmetricity scalar}\,,$$

$$C := 2(Q_{\kappa\rho\lambda}T^{\lambda\kappa\rho} + Q_{\rho}{}^{\sigma}{}_{\sigma}T^{\rho\kappa}{}_{\kappa} - Q^{\sigma}{}_{\sigma\rho}T^{\rho\kappa}{}_{\kappa})\,.$$

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$$S_{\rm TEGR}=\int\left[-\frac{1}{2\kappa^2}T+L_{\rm m}\right]e\,d^4x\,.$$

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- Then, these two theories are equivalent (classicality) to GR.

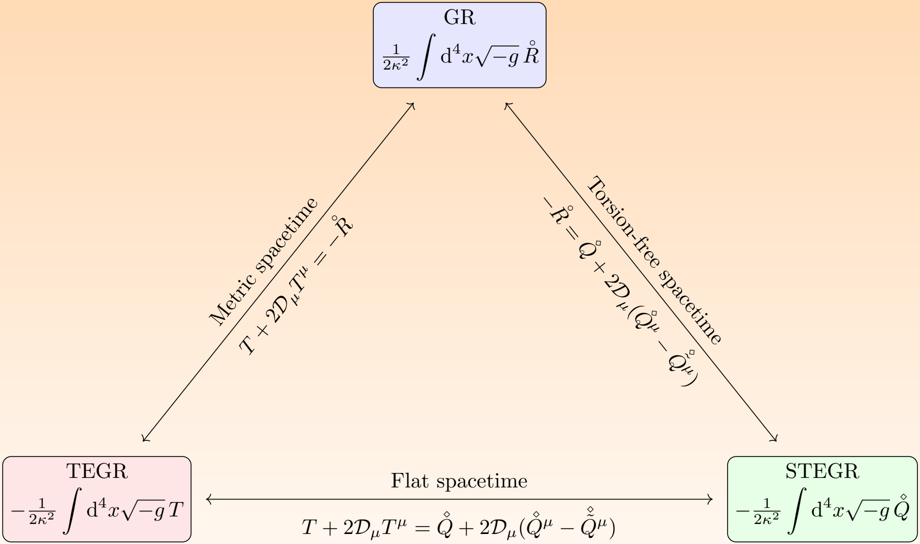

Figure: Geometrical trinity of gravity (S. Bahamonde et.al., “Teleparallel Gravity: From Theory o Cosmology,” Rept. Prog. Phys. **86** (2023) no.2, 026901.; J. Beltrán Jiménez, L. Heisenberg and T. S. Koivisto, “The Geometrical Trinity of Gravity,” Universe **5** (2019) no.7, 173.)

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$$\mathcal{L}_{\rm NGR} = c_1 T^{\mu\nu\rho} T_{\mu\nu\rho} + c_2 T^{\mu\nu\rho} T_{\rho\nu\mu} + c_3 T^{\mu}{}_{\mu\rho} T_{\nu}{}^{\nu\rho}\,.$$

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$$\mathcal{L}_{f(T)} = f(T)\,.$$

1. It has been highly analysed in cosmology - it gives interesting phenomenology.
2. However, it is strongly coupled in Minkowski and even in non-flat FLRW S. Bahamonde, K. F. Dialektopoulos, M. Hohmann, J. Levi Said, C. Pfeifer and E. N. Saridakis, Eur. Phys. J. C **83** (2023) no.3, 193

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- More theories to analyse such as Teleparallel Horndeksi where $G_5 \neq 0$ is avoided! S. Bahamonde, K. F. Dialektopoulos and J. Levi Said, Phys. Rev. D **100** (2019) no.6, 064018; S. Bahamonde, K. F. Dialektopoulos, V. Gakis and J. Levi Said, Phys. Rev. D **101** (2020) no.8, 084060

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Overview of the Talk

Gauge formalism of metric-affine geometry

- $A(4,\mathbb{R}) = T^4 \rtimes GL(4,\mathbb{R})$ gauge connection with an independent local metric structure[1]:

$$A_\mu = e^a{}_\mu P_a + \omega^a{}_{b\mu} L_a{}^b\,,$$
$$g_{\mu\nu} = e^a{}_\mu e^b{}_\nu g_{ab}\,,$$

where

$$\omega^a{}_{b\mu} = e^a{}_\lambda\, e_b{}^\rho\, \tilde{\Gamma}^\lambda{}_{\rho\mu} + e^a{}_\lambda\, \partial_\mu\, e_b{}^\lambda\,.$$

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- Generators of the affine group $A(4,\mathbb{R})$:

$$[P_a, P_b] = 0\,,$$
$$\left[L_a{}^b, P_c\right] = i\,\delta^b{}_c\, P_a\,,$$
$$\left[L_a{}^b, L_c{}^d\right] = i\big(\delta^b{}_c\, L_a{}^d - \delta_a{}^d\, L_c{}^b\big)\,.$$

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Poincaré gauge theory

- $ISO(1,3)$ translational and rotational field strength tensors:

$$F_{\mu\nu} = F^a{}_{\mu\nu} P_a + F^a{}_{b\mu\nu} J_a{}^b \,,$$

where

$$F^a{}_{\mu\nu} = \partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu + \omega^a{}_{b\mu}\, e^b{}_\nu - \omega^a{}_{b\nu}\, e^b{}_\mu \,,$$
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- Correspondence with the torsion and curvature tensors:

$$F^a{}_{\mu\nu} = e^a{}_\lambda T^\lambda{}_{\nu\mu} \,,$$
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$$\begin{aligned} G_{ab\mu} &= g_{ac}g_{bd}e^{c\lambda}e^{d\rho}Q_{\mu\lambda\rho}\,,\\ F^a{}_{\mu\nu} &= e^a{}_\lambda T^\lambda{}_{\nu\mu}\,,\\ F^a{}_{b\mu\nu} &= g_{bc}\,e^a{}_\lambda e^{c\rho}\tilde{R}^\lambda{}_{\rho\mu\nu}\,. \end{aligned}$$

Dynamics in MAG gauge theories

- Gravitational action with dynamical torsion and nonmetricity

$$S = \int d^4x\sqrt{-g}\Big[\mathcal{L}_m - \frac{1}{16\pi}\mathcal{L}_g\big(\tilde{\mathcal{R}}, \mathcal{T}, \mathcal{Q}\big)\Big]\,.$$

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- Correspondence between geometry and matter:

$$\begin{aligned}
\frac{1}{\sqrt{-g}}\frac{\delta\left(\mathcal{L}_g\sqrt{-g}\right)}{\delta e^a{}_\nu} &= 16\pi\theta_a{}^\nu\,,\\
\frac{1}{\sqrt{-g}}\frac{\delta\left(\mathcal{L}_g\sqrt{-g}\right)}{\delta \omega^a{}_{b\nu}} &= 16\pi\Delta_a{}^{b\nu}\,.
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- Let us assume that the metric, torsion and nonmetricity have the same cosmological symmetries (isotropy and homogeneity)

$$\mathcal{L}_\xi \bar{g}_{\mu\nu} = \mathcal{L}_\xi \bar{T}^{\lambda}{}_{\mu\nu} = \mathcal{L}_\xi \bar{Q}^{\lambda}{}_{\mu\nu} = 0\,.$$

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- By solving those equations one get the FLRW metric and torsion+nonmetricity satisfying homogeneity and isotropy:

$$\begin{aligned}
\bar{g} &= -\bar{n}_\mu \bar{n}_\nu \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu + \bar{P}_{\mu\nu} \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = -N^2 \mathrm{d}t \otimes \mathrm{d}t + a^2 \gamma_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j\,,\\
\bar{T}^\lambda{}_{\mu\nu} &= 2T_1(t)\,\bar{n}_{[\mu}\bar{P}_{\nu]}{}^\lambda + 2T_2(t)\,\bar{\varepsilon}^\lambda{}_{\mu\nu\rho}\bar{n}^\rho\,,\\
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where $\gamma_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j = \frac{\mathrm{d}r^2}{1-Kr^2} + r^2 \mathrm{d}\Omega^2$. Note that there are 5 independent functions coming from Post-Riemannian.

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- Our aim is to perform a $3+1$+SVT-decomposition for all the tensors in MAG up to linear perturbations to then write:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}\,,\quad \bar{T}^\lambda{}_{\mu\nu} = \bar{T}^\lambda{}_{\mu\nu} + \delta T^\lambda{}_{\mu\nu}\,,\quad \bar{Q}^\lambda{}_{\mu\nu} = \bar{Q}^\lambda{}_{\mu\nu} + \delta Q^\lambda{}_{\mu\nu}\,.$$

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- The majority of these studies predominantly focus on torsion.
- Usually, it is possible to avoid cosmological singularities and replace them a cosmic bounce.
- Dark energy can be explained by the scalar modes of torsion.
- It is possible to find inflationary models such as Einstein-Cartan couple to Higgs or other more complicated ones which are compatible with observations.

Decomposition into irreducible parts

Irreducible decomposition of the torsion tensor

$$T^{\lambda}{}_{\mu\nu} = \frac{1}{3}\left(\delta^{\lambda}{}_{\nu}T_{\mu} - \delta^{\lambda}{}_{\mu}T_{\nu}\right) + \frac{1}{6}\varepsilon^{\lambda}{}_{\rho\mu\nu}S^{\rho} + t^{\lambda}{}_{\mu\nu}\,.$$

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Irreducible decomposition of the nonmetricity tensor

$$Q_{\lambda\mu\nu} = g_{\mu\nu} W_{\lambda} + \not{Q}_{\lambda\mu\nu}\,,$$

$$\not{Q}_{\lambda\mu\nu} = \frac{1}{2}\left(g_{\lambda\mu}\Lambda_{\nu} + g_{\lambda\nu}\Lambda_{\mu}\right) - \frac{1}{4} g_{\mu\nu}\Lambda_{\lambda} + \frac{1}{3}\varepsilon_{\lambda\rho\sigma(\mu}\Omega_{\nu)}{}^{\rho\sigma} + q_{\lambda\mu\nu}\,.$$

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- Second tensor part $q_{\lambda\mu\nu} = Q_{(\lambda\mu\nu)} - g_{(\mu\nu} W_{\lambda)} - \frac{3}{4} g_{(\mu\nu}\Lambda_{\lambda)}$.

$3+1$ decomposition for FLRW - Torsion sector

- The irreducible modes of antisymmetric rank-3 tensor (as torsion) can be decomposed as

$$
\begin{aligned}
T_\mu &= \vec{T}_\mu - n_\mu \phi\,,\\
S_\mu &= \vec{\mathcal{S}}_\mu - n_\mu \varrho\,,\\
t_{\mu\nu\rho} &= 2n_{[\nu}\vec{A}_{\rho]\mu} + 2\Big(n_\mu n_{[\nu} + \frac{1}{2}P_{\mu[\nu}\Big)\vec{B}_{\rho]} + \frac{1}{2}\varepsilon_{\nu\rho}{}^{\alpha\beta}\Big[-n_\alpha \vec{\mathcal{A}}_{\beta\mu} + \Big(n_\beta n_\mu + \frac{1}{2}P_{\beta\mu}\Big)\vec{\mathcal{B}}_\alpha\Big]\,.
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\end{aligned}
$$

- Then, torsion is decomposed as

$$
T^\lambda{}_{\mu\nu} = \bar{T}^\lambda{}_{\mu\nu} + \delta T^\lambda{}_{\mu\nu} = \begin{cases}
T^0{}_{0i} = -\dfrac{a}{3}(\vec{T}_i + 3\vec{B}_i)\\
T^0{}_{ij} = -\dfrac{a^2}{6N}\varepsilon_{ijk}(\vec{\mathcal{S}}^k + 3\vec{\mathcal{B}}^k)\\
T^i{}_{0j} = -N\Big[\vec{A}^i{}_j - \dfrac{1}{3}\delta^i{}_j\,(\bar{\phi}+\phi) + \dfrac{1}{6}\varepsilon^i{}_{jk}\Big(\vec{\mathcal{S}}^k - \dfrac{3}{2}\vec{\mathcal{B}}^k\Big)\Big]\\
T^i{}_{jk} = -\dfrac{a}{6}\Big\{4\delta^i{}_{[j}\vec{T}_{k]} - 6\delta^i{}_{[j}\vec{B}_{k]} + \Big[3\varepsilon_{jkl}\vec{\mathcal{A}}^{il} - \varepsilon^i{}_{jk}\,(\bar{\varrho}+\varrho)\Big]\Big\}
\end{cases},
$$

where we have introduced the background T_i in the scalars ϕ, ϱ.

$3+1$ decomposition for FLRW - Nonmetricity sector

- The irreducible modes of symmetric rank-3 tensor (as nonmetricity) can be decomposed as

$$
\begin{aligned}
W_\mu &= \vec{W}_\mu - n_\mu \theta \,, \\
\Lambda_\mu &= \vec{\Lambda}_\mu - n_\mu \psi \,, \\
\Omega_{\mu\nu\rho} &= 2 n_{[\nu} \vec{\mathcal{Q}}_{\rho]\mu} + 2\Big(n_\mu n_{[\nu} + \frac{1}{2} P_{\mu[\nu}\Big) \vec{\mathcal{Y}}_{\rho]} + \frac{1}{2} \varepsilon_{\nu\rho}{}^{\alpha\beta}\Big[- n_\alpha \vec{Q}_{\beta\mu} + \Big(n_\beta n_\mu + \frac{1}{2} P_{\beta\mu}\Big)\vec{Y}_\alpha\Big] \,, \\
q_{\mu\nu\rho} &= \vec{C}_{\mu\nu\rho} - 3 n_{(\mu} \vec{\kappa}_{\nu\rho)} + \frac{3}{5}\left(5 n_{(\mu} n_\nu + P_{(\mu\nu}\right)\vec{Z}_{\rho)} - \left(n_{(\mu} n_\nu + P_{(\mu\nu}\right) n_{\rho)} \xi \,.
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q_{\mu\nu\rho} &= \vec{C}_{\mu\nu\rho} - 3n_{(\mu}\vec{\kappa}_{\nu\rho)} + \frac{3}{5}\big(5n_{(\mu}n_\nu + P_{(\mu\nu)}\big)\vec{Z}_{\rho)} - \big(n_{(\mu}n_\nu + P_{(\mu\nu)}\big)\, n_{\rho)}\xi \,.
\end{aligned}
$$

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$$
Q^\lambda{}_{\mu\nu} = \begin{cases}
Q_{000} = -N^3\Big[\bar{\theta} + \theta + \frac{3}{4}\left(\bar{\psi} + \psi\right) - \left(\bar{\xi} + \xi\right)\Big] \\
Q_{00i} = -\frac{N^2 a}{2}\Big(\vec{\Lambda}_i - \frac{1}{3}\vec{Y}_i - 2\vec{Z}_i\Big) \\
Q_{0ij} = -\frac{N a^2}{3}\Big\{\vec{Q}_{ij} - 3\vec{\kappa}_{ij} - 3\Big[\bar{\theta} + \theta - \frac{1}{4}\left(\bar{\psi} + \psi\right) + \frac{1}{3}\left(\bar{\xi} + \xi\right)\Big]\gamma_{ij}\Big\} \\
Q_{i00} = -N^2 a\Big(\vec{W}_i - \frac{1}{4}\vec{\Lambda}_i + \frac{1}{3}\vec{Y}_i - \vec{Z}_i\Big) \\
Q_{i0j} = \frac{N a^2}{6}\Big\{[\vec{Q}_{ij} + 6\vec{\kappa}_{ij} + (3\left(\bar{\psi} + \psi\right) + 2\left(\bar{\xi} + \xi\right))\gamma_{ij}] - 3\varepsilon_{ijk}\vec{\mathcal{Y}}^k\Big\} \\
Q_{ijk} = a^3\Big\{\vec{C}_{ijk} + \frac{1}{4}(4\vec{W}_i - \vec{\Lambda}_i)\gamma_{jk} + \gamma_{i(j}\vec{\Lambda}_{k)} - \frac{1}{6}(\gamma_{jk}\vec{Y}_i - \gamma_{i(j}\vec{Y}_{k)}) + \frac{3}{5}\gamma_{(ij}\vec{Z}_{k)} + \frac{2}{3}\varepsilon_{li(j}\vec{\mathcal{Q}}^l{}_{k)}
\end{cases}
$$

where we have introduced the background Q_i in the scalars θ, ψ, ξ.

Summary of modes

- We can group the modes as:

$$\delta\vec{\mathbf{X}} = \{\alpha,\zeta,\phi,\varrho,\theta,\psi,\xi\}\,,$$
$$\delta\vec{\mathbf{X}}_i = \{\vec{\beta}_i,\vec{T}_i,\vec{\mathcal{S}}_i,\vec{B}_i,\vec{\mathcal{B}}_i,\vec{W}_i,\vec{\Lambda}_i,\vec{Y}_i,\vec{\mathcal{Y}}_i,\vec{Z}_i\}\,,$$
$$\delta\vec{\mathbf{X}}_{ij} = \{\vec{h}_{ij},\vec{A}_{ij},\vec{\mathcal{A}}_{ij},\vec{Q}_{ij},\vec{\mathcal{Q}}_{ij},\vec{\kappa}_{ij}\}\,,$$
$$\delta\vec{\mathbf{X}}_{ijk} = \{\vec{C}_{ijk}\}\,.$$

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Spin and Parity	$\mathbf{0}^+$	$\mathbf{0}^-$	$\mathbf{1}^+$	$\mathbf{1}^-$	$\mathbf{2}^+$	$\mathbf{2}^-$	$\mathbf{3}^-$
Metric sector $g_{\mu\nu}$	α, ζ	-	-	$\vec{\beta}_i$	$\vec{h}_{ij}$	-	-
Torsion sector $T^\lambda{}_{\mu\nu}$	ϕ	ϱ	$\vec{\mathcal{S}}_i, \vec{\mathcal{B}}_i$	$\vec{T}_i, \vec{B}_i$	$\vec{A}_{ij}$	$\vec{\mathcal{A}}_{ij}$	-
Nonmetricity sector $Q_{\lambda\mu\nu}$	θ, ψ, ξ	-	$\vec{\mathcal{Y}}_i$	$\vec{W}_i, \vec{\Lambda}_i, \vec{Y}_i, \vec{Z}_i$	$\vec{Q}_{ij}, \vec{\kappa}_{ij}$	$\vec{\mathcal{Q}}_{ij}$	$\vec{C}_{ijk}$

Table: Species in MAG. K. Aoki, S. Bahamonde, J. Gigante Valcarcel and M. A. Gorji, "Cosmological Perturbation Theory in Metric-Affine Gravity," Phys. Rev. D **110** (2024) no.2, 2.

Helicity (SVT) decomposition

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- After the $3+1$ decomposition, we only need to deal with the spatial scalar and tensors with Latin indices belonging to the maximally symmetric space γ_{ij} (and with the "external" parameter t).
- Let us decompose them into different helicity sectors:

$$\delta\vec{\mathbf{X}}_i = \mathrm{D}_i\mathbf{S} + \mathbf{V}_i^{(\mathrm{T})}\,,$$
$$\delta\vec{\mathbf{X}}_{ij} = \Big(\mathrm{D}_{(i}\mathrm{D}_{j)} - \frac{1}{3}\gamma_{ij}\mathrm{D}^2\Big)\mathbf{S} + \mathrm{D}_{(i}\mathbf{V}_{j)}^{(\mathrm{T})} + \mathbf{T}_{ij}^{(\mathrm{TT})}\,,$$
$$\delta\vec{\mathbf{X}}_{ijk} = \Big[\mathrm{D}_{(i}\mathrm{D}_j\mathrm{D}_{k)} - \frac{1}{5}\gamma_{(ij}\mathrm{D}_{k)}\big(3\mathrm{D}^2+4K\big)\Big]\mathbf{S} + \Big[\mathrm{D}_{(i}\mathrm{D}_j - \frac{1}{5}\gamma_{(ij}\big(\mathrm{D}^2+2K\big)\Big]\mathbf{V}_{k)}^{(\mathrm{T})} + \mathrm{D}_{(i}\mathbf{T}_{jk)}^{(\mathrm{TT}}$$

Here, the superscript "(T)" refers that the quantity is transverse, while the tensors denoted by the superscript "(TT)" are transverse-traceless and symmetric:

$$\mathrm{D}^i\mathbf{V}_i^{(\mathrm{T})} = 0\,,$$
$$\mathrm{D}^i\mathbf{T}_{ij}^{(\mathrm{TT})} = \mathbf{T}_{ij}^{(\mathrm{TT})}\gamma^{ij} = 0\,,\quad \mathbf{T}_{ij}^{(\mathrm{TT})} = \mathbf{T}_{(ij)}^{(\mathrm{TT})}\,,$$
$$\mathrm{D}^i\mathbf{TT}_{ijk}^{(\mathrm{TT})} = \mathbf{TT}_{ijk}^{(\mathrm{TT})}\gamma^{ij} = 0\,,\quad \mathbf{TT}_{ijk}^{(\mathrm{TT})} = \mathbf{TT}_{(ijk)}^{(\mathrm{TT})}\,.$$

SVT decomposition around FRW

- Thereby, the 10 dof described by the metric perturbations are split in terms of four scalars $\{\alpha, \beta, \zeta, h\}$ (1 dof each), two transverse vectors $\{\beta_i^{(\mathrm{T})}, h_i^{(\mathrm{T})}\}$ (2 dof each), and one symmetric and transverse-traceless tensor $h_{ij}^{(\mathrm{TT})}$ (2 dof).

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-

SVT	Quantities	dof	Total dof
5 scalars	$\{T, B, \phi, A, \varrho\}$	1 dof each	5
3 pseudoscalars	$\{\mathcal{S}, \mathcal{B}, \mathcal{A}\}$	1 dof each	3
3 vectors	$\{T_i^{(\mathrm{T})}, B_i^{(\mathrm{T})}, A_i^{(\mathrm{T})}\}$	2 dof each	6
3 pseudovectors	$\{\mathcal{S}_i^{(\mathrm{T})}, \mathcal{B}_i^{(\mathrm{T})}, \mathcal{A}_i^{(\mathrm{T})}\}$	2 dof each	6
1 rank-2 tensor	$\{A_{ij}^{(\mathrm{TT})}\}$	2 dof each	2
1 rank-2 pseudotensor	$\{\mathcal{A}_{ij}^{(\mathrm{TT})}\}$	2 dof each	2

Table: Perturbation spectrum for the torsion tensor. K. Aoki, S. Bahamonde, J. Gigante Valcarcel and M. A. Gorji, "Cosmological Perturbation Theory in Metric-Affine Gravity," Phys. Rev. D **110** (2024) no.2, 2.

SVT decomposition around FRW

SVT	Quantities	dof	Total dof
10 scalars	$\{\theta, \psi, \xi, \Lambda, Y, Z, \kappa, Q, W, C\}$	1 dof each	10
2 pseudoscalars	$\{\mathcal{Y}, \mathcal{Q}\}$	1 dof each	2
7 vectors	$\{\Lambda^{(\mathrm{T})}{}_i, Y^{(\mathrm{T})}{}_i, Z^{(\mathrm{T})}{}_i, \kappa^{(\mathrm{T})}{}_i, Q^{(\mathrm{T})}{}_i, W^{(\mathrm{T})}{}_i, C^{(\mathrm{T})}{}_i\}$	2 dof each	14
2 pseudovectors	$\{\mathcal{Y}^{(\mathrm{T})}{}_i, \mathcal{Q}^{(\mathrm{T})}{}_i\}$	2 dof each	4
3 rank-2 tensor	$\{\kappa^{(\mathrm{TT})}{}_{ij}, Q^{(\mathrm{TT})}{}_{ij}, C^{(\mathrm{TT})}{}_{ij}\}$	2 dof each	6
1 rank-2 pseudotensor	$\{\mathcal{Q}^{(\mathrm{TT})}{}_{ij}\}$	2 dof each	2
1 rank-3 tensor	$\{C^{(\mathrm{TT})}{}_{ijk}\}$	2 dof each	2

Table: Perturbation spectrum for the nonmetricity tensor.

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Table: Perturbation spectrum for the nonmetricity tensor.

- Note that the spin-3 carries a helicity-3 part and sice the metric does not have such terms, it totally decouples from the other modes in FLRW.

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- Note that the spin-3 carries a helicity-3 part and sice the metric does not have such terms, it totally decouples from the other modes in FLRW.
- We studied this spin-3 part finding

$$\ddot{C}^{(A)} + 3H\dot{C}^{(A)} + [\omega^{(A)}_{C,k}]^2 C^{(A)} = J^{(A)}\,.$$

Overview of the Talk

1. Brief introduction to metric-affine geometry
2. Teleparallel theories
   - Trinity of Gravity
   - Modified Theories with torsion and applications (Metric TG)
   - Theories with Nonmetricity and applications (Symmetric TG)
3. Gauge theories of gravity
4. Cosmological perturbations in MAG
5. Algebraic Classification in MAG
6. Stability in Metric Affine Gauge theories
   - Stability issues in quadratic Poincaré gauge theory
   - Cubic extensions
7. Conclusions and future prospects

How is the procedure for algebraic classification?

- The complete classification of the relevant tensors in Riemannian geometry is known.

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Riemmanian Curvature decomposition

$$R_{\lambda\rho\mu\nu} = W_{\lambda\rho\mu\nu} + \frac{1}{2}\left(g_{\lambda\mu}\not{R}_{\rho\nu} + g_{\rho\nu}\not{R}_{\lambda\mu} - g_{\lambda\nu}\not{R}_{\rho\mu} - g_{\rho\mu}\not{R}_{\lambda\nu}\right) + \frac{1}{6}\,R\,g_{\lambda[\mu}g_{\nu]\rho}\,,$$

$$\#20(R_{\lambda\rho\mu\nu}) = \#10(W_{\lambda\rho\mu\nu}) + \#9(\not{R}_{\rho\nu}) + \#1(R)\,.$$

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- **Result in Riemannian geometry:** Weyl has 6 types **(Petrov classification)**; Ricci traceless has 15 types**(Segre classification)**;
- **What happens in GR in vacuum?** $\not{R}_{\rho\nu} = R = 0$ and then the curvature is fully characterised by the Weyl tensor with their 6 types.

- This classification can be derived by means of its principal null directions which requires expressing any tensor in terms of a set of null vectors l_μ, k_μ, m_μ, and $\bar{m}_\mu$;

$$\begin{aligned} k^\mu l_\mu &= -m^\mu \bar{m}_\mu = 1\,, \\ k^\mu m_\mu &= k^\mu \bar{m}_\mu = l^\mu m_\mu = l^\mu \bar{m}_\mu = 0\,, \\ k^\mu k_\mu &= l^\mu l_\mu = m^\mu m_\mu = \bar{m}^\mu \bar{m}_\mu = 0\,. \end{aligned}$$

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Algebraic type	Segre characteristic	Intrinsic characterisation
I	$[1\,1\,1]$	$l_{[\sigma}\,{}^{(1)}\tilde{W}_{\lambda]\rho\mu[\nu}l_{\omega]}l^\rho l^\mu = 0$
II	$[2\,1]$	${}^{(1)}\tilde{W}_{\lambda\rho\mu[\nu}l_{\omega]}l^\rho l^\mu = 0$
D	$[(1\,1)\,1]$	${}^{(1)}\tilde{W}_{\lambda\rho\mu[\nu}k_{\omega]}k^\rho k^\mu = {}^{(1)}\tilde{W}_{\lambda\rho\mu[\nu}l_{\omega]}l^\rho l^\mu = 0$
III	$[3]$	${}^{(1)}\tilde{W}_{\lambda\rho\mu[\nu}l_{\omega]}l^\mu = 0$
N	$[(2\,1)]$	${}^{(1)}\tilde{W}_{\lambda\rho\mu\nu}l^\mu = 0$
O	$[-]$	${}^{(1)}\tilde{W}_{\lambda\rho\mu\nu} = 0$

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- Reissner-Nordström and also Kerr-Newman have a very particular characterisation which is known as Type D. Not only the Weyl tensor is Type D in those cases but also the Faraday tensor fulfills a similar property.
- The most general Type D solution in Einstein-Maxwell is known as the Plebański-Demiański characterised by $\{M, a, \alpha, N\}$ (mass, angular momentum, acceleration and Nut charge) and the electromagnetic charges.

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- Reissner-Nordström and also Kerr-Newman have a very particular characterisation which is known as Type D. Not only the Weyl tensor is Type D in those cases but also the Faraday tensor fulfills a similar property.
- The most general Type D solution in Einstein-Maxwell is known as the Plebański-Demiański characterised by $\{M, a, \alpha, N\}$ (mass, angular momentum, acceleration and Nut charge) and the electromagnetic charges.
- *Goldberg-Sachs theorem:* A vacuum solution of the Einstein's field equations admits a shear-free null geodesic congruence if and only if the conformal part of the Riemann tensor is algebraically special.

Curvature decomposition in Metric-Affine geometry

- First, we need to find the building blocks of the general curvature tensor (recall that in Riemannian geometry, we had 3: $W_{\alpha\beta\mu\nu}$, $\tilde{\not R}_{\mu\nu}$, and R).

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$$\tilde{R}_{\lambda\rho\mu\nu} = \tilde{R}_{[\lambda\rho]\mu\nu} + \tilde{R}_{(\lambda\rho)\mu\nu} := \tilde{W}_{\lambda\rho\mu\nu} + \tilde{Z}_{\lambda\rho\mu\nu}\,.$$

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- We find that there are 11 building blocks S. Bahamonde and J. Gigante Valcarcel, Phys. Rev. D **108** (2023) no.4, 4:

Building block	Number of independent components	Limit in Riemannian geometry
${}^{(1)}\tilde{Z}_{\lambda\rho\mu\nu}$	30	zero
${}^{(1)}\tilde{W}_{\lambda\rho\mu\nu}$	10	Weyl tensor $W_{\lambda\rho\mu\nu}$
$\tilde{\not{R}}^{(T)}_{\lambda[\rho\mu\nu]}$	9	zero
$\tilde{\not{R}}^{(Q)}_{\lambda[\rho\mu\nu]}$	9	zero
$\tilde{\not{R}}_{(\mu\nu)}$	9	Ricci traceless $\not{R}_{\mu\nu}$
$\hat{\not{R}}^{(Q)}_{(\mu\nu)}$	9	zero
$\tilde{R}^{(T)}_{[\mu\nu]}$	6	zero
$\hat{R}^{(Q)}_{[\mu\nu]}$	6	zero
$\tilde{R}^{\lambda}{}_{\lambda\mu\nu}$	6	zero
$\tilde{R}$	1	Ricci scalar R
$*\tilde{R}$	1	zero

Algebraic classification in MAG

- It turns out that the sets $\{\tilde{\not{R}}^{(T)}_{\lambda[\rho\mu\nu]}, \tilde{\not{R}}^{(Q)}_{\lambda[\rho\mu\nu]}, \tilde{\not{R}}_{(\mu\nu)}, \hat{\not{R}}^{(Q)}_{(\mu\nu)}\}$ and $\{\tilde{R}^{(T)}_{[\mu\nu]}, \hat{R}^{(Q)}_{[\mu\nu]}, \tilde{R}^{\lambda}{}_{\lambda\mu\nu}\}$ contain building blocks with 9 and 6 independent components.

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$$ {}^{(1)}\tilde{Z}^{\lambda}{}_{\lambda\mu\nu} = {}^{(1)}\tilde{Z}^{\lambda}{}_{\mu\lambda\nu} = 0\,, \quad {}^{(1)}\tilde{Z}_{\lambda[\rho\mu\nu]} = 0\,. $$

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  3. It is common that in spherical symmetry, the field strength tensors are of Type D (two null directions aligned). For a black hole solution endowed with shears, we found that even in spherical symmetry, ${}^{(1)}\tilde{Z}^{\lambda}{}_{\lambda\mu\nu}$ is no longer Type D.

Type[2]	Main case	Exceptional cases	Complex scalars	Intrinsic characterisation
I	(I,I)	$(\mathrm{I},\mathrm{I}_\infty)$	$\Delta_0=\Delta_{14}=0$	${}^{(1)}\tilde{Z}_{\lambda\rho\mu[\nu}l_{\sigma]}l^\lambda l^\rho l^\mu=0$ ${}^{(1)}\tilde{Z}_{\lambda\rho\mu[\nu}k_{\sigma]}k^\lambda k^\rho k^\mu=0$
C	(II,I)	$(\mathrm{II},\mathrm{I}_\infty)$	$\{\Delta_i\}^{i=0,\cdots,2}=\Delta_{14}=0$	$l_{[\omega}{}^{(1)}\tilde{Z}_{\lambda]\rho\mu[\nu}l_{\sigma]}l^\rho l^\mu=0$ ${}^{(1)}\tilde{Z}_{\lambda\rho\mu[\nu}k_{\sigma]}k^\lambda k^\rho k^\mu=0$
C*	$(\mathrm{II},-)$	-	e.g. only $\Delta_3,\Delta_{14}\neq 0$ and $\Delta_3/\Delta_{14}\in\mathbb{R}^-$	$l_{[\omega}{}^{(1)}\tilde{Z}_{\lambda]\rho\mu[\nu}l_{\sigma]}l^\rho l^\mu=0$ ${}^{(1)}\tilde{Z}_{\lambda\rho\mu[\nu}k_{\sigma]}k^\lambda k^\rho k^\mu\neq 0\;\forall\,k^\mu\neq l^\mu$
B	$(\mathrm{II},\mathrm{II})$	$(\mathrm{II},\mathrm{II}_\infty)$ $(\mathrm{II},\mathrm{II})_{\mathrm{I}_\infty}$	$\{\Delta_i\}^{i=0,\cdots,2}=\{\Delta_i\}^{i=12,\cdots,14}=0$	$l_{[\omega}{}^{(1)}\tilde{Z}_{\lambda]\rho\mu[\nu}l_{\sigma]}l^\rho l^\mu=0$ $k_{[\omega}{}^{(1)}\tilde{Z}_{\lambda]\rho\mu[\nu}k_{\sigma]}k^\rho k^\mu=0$
S	$(\mathrm{II},\mathrm{II},\mathrm{II})$		$\{\Delta_i\}^{i=0,\cdots,2}=\{\Delta_i\}^{i=12,\cdots,14}=0$ and special cases	$l_{[\omega}{}^{(1)}\tilde{Z}_{\lambda]\rho\mu[\nu}l_{\sigma]}l^\rho l^\mu=0$ $k_{[\omega}{}^{(1)}\tilde{Z}_{\lambda]\rho\mu[\nu}k_{\sigma]}k^\rho k^\mu=0$ $l'_{[\omega}{}^{(1)}\tilde{Z}_{\lambda]\rho\mu[\nu}l'_{\sigma]}l'^\rho l'^\mu=0$
K	$(\mathrm{III},\mathrm{I})$	$(\mathrm{III},\mathrm{I}_\infty)$	$\{\Delta_i\}^{i=0,\cdots,5}=\Delta_{14}=0$	$l^{[\tau}l_{[\omega}{}^{(1)}\tilde{Z}^{\lambda]}{}_{\rho]\mu[\nu}l_{\sigma]}l^\mu=0$ ${}^{(1)}\tilde{Z}_{\lambda\rho\mu[\nu}k_{\sigma]}k^\lambda k^\rho k^\mu=0$
K*	$(\mathrm{III},-)$	-	$\{\Delta_i\}^{i=0,\cdots,5}=0$ and e.g. $\{\Delta_i\}^{i=8,\cdots,13}=0\,,\,\Delta_6=3\Delta_7,\ \Delta_6\bar{\Delta}_{14}\notin\mathbb{R}$	$l^{[\tau}l_{[\omega}{}^{(1)}\tilde{Z}^{\lambda]}{}_{\rho]\mu[\nu}l_{\sigma]}l^\mu=0$ ${}^{(1)}\tilde{Z}_{\lambda\rho\mu[\nu}k_{\sigma]}k^\lambda k^\rho k^\mu\neq 0\;\forall\,k^\mu\neq l^\mu$
M	$(\mathrm{III},\mathrm{II})$	$(\mathrm{III},\mathrm{II}_\infty)$ $(\mathrm{III},\mathrm{II})_{\mathrm{I}_\infty}$	$\{\Delta_i\}^{i=0,\cdots,5}=\{\Delta_i\}^{i=12,\cdots,14}=0$	$l^{[\tau}l_{[\omega}{}^{(1)}\tilde{Z}^{\lambda]}{}_{\rho]\mu[\nu}l_{\sigma]}l^\mu=0$ $k_{[\omega}{}^{(1)}\tilde{Z}_{\lambda]\rho\mu[\nu}k_{\sigma]}k^\rho k^\mu=0$
D	$(\mathrm{III},\mathrm{III})$	$(\mathrm{III},\mathrm{III})_{\mathrm{II}_\infty}$ $(\mathrm{III},\mathrm{III})_{\mathrm{I}_\infty}$	$\{\Delta_i\}^{i=0,\cdots,5}=\{\Delta_i\}^{i=9,\cdots,14}=0$	$l^{[\tau}l_{[\omega}{}^{(1)}\tilde{Z}^{\lambda]}{}_{\rho]\mu[\nu}l_{\sigma]}l^\mu=0$ $k^{[\tau}k_{[\omega}{}^{(1)}\tilde{Z}^{\lambda]}{}_{\rho]\mu[\nu}k_{\sigma]}k^\mu=0$
H	(IV,I)	$(\mathrm{IV},\mathrm{I}_\infty)$	$\{\Delta_i\}^{i=0,\cdots,8}=\Delta_{14}=0$	$l_{[\omega}{}^{(1)}\tilde{Z}_{\lambda]\rho\mu[\nu}l_{\sigma]}l^\mu=0$ $l_\lambda l_{[\omega}{}^{(1)}\tilde{Z}^{\lambda}{}_{\rho]\mu\nu}=0$ ${}^{(1)}\tilde{Z}_{\lambda\rho\mu[\nu}k_{\sigma]}k^\lambda k^\rho k^\mu=0$
H*	$(\mathrm{IV},-)$	-	$\{\Delta_i\}^{i=0,\cdots,8}=0$ and e.g. $\Delta_9=\Delta_{11}=\Delta_{12}=\Delta_{13}=0\,,\,\Delta_{10}\bar{\Delta}_{14}\notin\mathbb{R}$	$l_{[\omega}{}^{(1)}\tilde{Z}_{\lambda]\rho\mu[\nu}l_{\sigma]}l^\mu=0$ $l_\lambda l_{[\omega}{}^{(1)}\tilde{Z}^{\lambda}{}_{\rho]\mu\nu}=0$ ${}^{(1)}\tilde{Z}_{\lambda\rho\mu[\nu}k_{\sigma]}k^\lambda k^\rho k^\mu\neq 0\;\forall\,k^\mu\neq l^\mu$
F	$(\mathrm{IV},\mathrm{II})$	$(\mathrm{IV},\mathrm{II}_\infty)$ $(\mathrm{IV},\mathrm{II})_{\mathrm{I}_\infty}$	$\{\Delta_i\}^{i=0,\cdots,8}=\{\Delta_i\}^{i=12,\cdots,14}=0$	$l_{[\omega}{}^{(1)}\tilde{Z}_{\lambda]\rho\mu[\nu}l_{\sigma]}l^\mu=0$ $l_\lambda l_{[\omega}{}^{(1)}\tilde{Z}^{\lambda}{}_{\rho]\mu\nu}=0$ $k_{[\omega}{}^{(1)}\tilde{Z}_{\lambda]\rho\mu[\nu}k_{\sigma]}k^\rho k^\mu=0$
L	(V,I)	$(\mathrm{V},\mathrm{I}_\infty)$	$\{\Delta_i\}^{i=0,\cdots,11}=\Delta_{14}=0$	$l^{[\tau}l_{[\omega}{}^{(1)}\tilde{Z}^{\lambda]}{}_{\rho]\mu\nu}=0$ ${}^{(1)}\tilde{Z}_{\lambda\rho\mu[\nu}k_{\sigma]}k^\lambda k^\rho k^\mu=0$
L*	$(\mathrm{V},-)$	-	$\{\Delta_i\}^{i=0,\cdots,11}=0$ $\Delta_{14}\bar{\Delta}_{12}-2\Delta_{13}\bar{\Delta}_{14}=0,\ $ and $\Delta_{12}\neq\bar{\Delta}_{12}$	$l^{[\tau}l_{[\omega}{}^{(1)}\tilde{Z}^{\lambda]}{}_{\rho]\mu\nu}=0$ ${}^{(1)}\tilde{Z}_{\lambda\rho\mu[\nu}k_{\sigma]}k^\lambda k^\rho k^\mu\neq 0\;\forall\,k^\mu\neq l^\mu$
N	$(\mathrm{VI},-)$	-	$\{\Delta_i\}^{i=0,\cdots,13}=0$	$l_{[\sigma}{}^{(1)}\tilde{Z}_{\lambda]\rho\mu\nu}=0$
O	-	-	$\{\Delta_i\}^{i=0,\cdots,14}=0$	${}^{(1)}\tilde{Z}_{\lambda\rho\mu\nu}=0$

[2] S. Bahamonde, J. Gigante Valcarcel and J. M. M. Senovilla, “Algebraic classification of the gravitational field in general metric-affine geometries,” arXiv:2409.07153

Overview of the Talk

- Class of quadratic Poincaré gauge models that are reduced to General Relativity in the absence of torsion:

$$S_g = \frac{1}{16\pi}\int\Big[c_2\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\lambda\mu\rho\nu} - R - \frac{1}{2}\left(2c_1 + c_2\right)\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\mu\nu\lambda\rho} + c_1\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\lambda\rho\mu\nu} + d_1\tilde{R}_{\mu\nu}\big(\tilde{R}^{\mu\nu} - \tilde{R}^{\nu\mu}\big) + \frac{1}{2}\big(m_T^2 T_\mu T^\mu + m_S^2 S_\mu S^\mu + m_t^2 t_{\lambda\mu\nu}t^{\lambda\mu\nu}\big)\Big]\sqrt{-g}\,d^4x\,.$$

Vector and axial sectors in quadratic Poincaré gauge theory

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$$+ d_1\tilde{R}_{\mu\nu}\big(\tilde{R}^{\mu\nu}-\tilde{R}^{\nu\mu}\big) + \frac{1}{2}\big(m_T^2 T_\mu T^\mu + m_S^2 S_\mu S^\mu + m_t^2 t_{\lambda\mu\nu}t^{\lambda\mu\nu}\big)\Big]\sqrt{-g}\,d^4x\,.$$

- Focusing on the vector and axial modes of torsion:

$$\tilde{R}_{\lambda\rho\mu\nu} = R_{\lambda\rho\mu\nu} + \frac{2}{3}\left(g_{\lambda[\nu}\nabla_{\mu]}T_\rho - g_{\rho[\nu}\nabla_{\mu]}T_\lambda\right) + \frac{1}{6}\varepsilon_{\lambda\rho\sigma[\mu}\nabla_{\nu]}S^\sigma$$
$$+ \frac{2}{9}\left(2T_{[\lambda}g_{\rho][\nu}T_{\mu]} + g_{\lambda[\nu}g_{\mu]\rho}T_\sigma T^\sigma\right) + \frac{1}{72}\left(2S_{[\lambda}g_{\rho][\mu}S_{\nu]} + g_{\lambda[\mu}g_{\nu]\rho}S_\sigma S^\sigma\right)$$
$$+ \frac{1}{18}\left[2\varepsilon_{\mu\nu\sigma[\lambda}T_{\rho]}S^\sigma + \left(g_{\lambda[\nu}\varepsilon_{\mu]\rho\sigma\omega} - g_{\rho[\nu}\varepsilon_{\mu]\lambda\sigma\omega}\right)T^\sigma S^\omega\right]\,,$$
$$\tilde{R}_{\mu\nu} = R_{\mu\nu} - \frac{1}{3}\left(2\nabla_\nu T_\mu + g_{\mu\nu}\nabla_\lambda T^\lambda\right) + \frac{1}{12}\varepsilon_{\lambda\rho\mu\nu}\nabla^\lambda S^\rho$$
$$+ \frac{2}{9}\left(T_\mu T_\nu - g_{\mu\nu}T_\lambda T^\lambda\right) + \frac{1}{72}\left(g_{\mu\nu}S_\lambda S^\lambda - S_\mu S_\nu\right)\,.$$

Vector and axial sectors in quadratic Poincaré gauge theory

- Instabilities in the vector and axial sectors:

$$
\begin{aligned}
\mathcal{L}_g^{(2)} \propto\ & \frac{1}{9}\left(4c_1+c_2+2d_1\right)F^{(T)}_{\mu\nu}F^{(T)\mu\nu}-\frac{1}{72}\left(4c_1+c_2+d_1\right)F^{(S)}_{\mu\nu}F^{(S)\mu\nu}\\
&+\frac{1}{24}\left(c_2-2c_1\right)\nabla_\mu S^\mu\nabla_\nu S^\nu+\frac{1}{27}\left(c_1-2c_2\right)S^\mu S^\nu\nabla_\mu T_\nu\\
&-\frac{1}{108}\left(4c_1+c_2\right)S_\mu S^\mu\nabla_\nu T^\nu+\frac{1}{18}\left(2c_1-c_2\right)T^\mu S^\nu\nabla_\nu S_\mu\\
&+\frac{1}{36}\left(4c_1+c_2\right)G_{\mu\nu}S^\mu S^\nu+\frac{1}{72}\left(4c_1+c_2\right)RS_\mu S^\mu\\
&-\frac{1}{162}\left(4c_1+c_2\right)T_\mu T^\mu S_\nu S^\nu+\frac{1}{81}\left(2c_2-c_1\right)T_\mu S^\mu T_\nu S^\nu\\
&+\frac{1}{2}m_T^2T_\mu T^\mu+\frac{1}{2}m_S^2S_\mu S^\mu\,.
\end{aligned}
$$

and then $c_1=c_2=d_1=0$ (only the masses survives in general backgrounds!)

- Instabilities in the vector and axial sectors:

$$\begin{aligned}\mathcal{L}_g^{(2)} \propto\ & \frac{1}{9}(4c_1+c_2+2d_1)\,F^{(T)}_{\mu\nu}F^{(T)\mu\nu} - \frac{1}{72}(4c_1+c_2+d_1)\,F^{(S)}_{\mu\nu}F^{(S)\mu\nu} \\ & + \frac{1}{24}(c_2-2c_1)\,\nabla_\mu S^\mu \nabla_\nu S^\nu + \frac{1}{27}(c_1-2c_2)\,S^\mu S^\nu \nabla_\mu T_\nu \\ & - \frac{1}{108}(4c_1+c_2)\,S_\mu S^\mu \nabla_\nu T^\nu + \frac{1}{18}(2c_1-c_2)\,T^\mu S^\nu \nabla_\nu S_\mu \\ & + \frac{1}{36}(4c_1+c_2)\,G_{\mu\nu}S^\mu S^\nu + \frac{1}{72}(4c_1+c_2)\,R S_\mu S^\mu \\ & - \frac{1}{162}(4c_1+c_2)\,T_\mu T^\mu S_\nu S^\nu + \frac{1}{81}(2c_2-c_1)\,T_\mu S^\mu T_\nu S^\nu \\ & + \frac{1}{2}m_T^2 T_\mu T^\mu + \frac{1}{2}m_S^2 S_\mu S^\mu\,.\end{aligned}$$

and then $c_1 = c_2 = d_1 = 0$ (only the masses survives in general backgrounds!)

Important problem!

It is not possible to have a stable propagating torsion tensor in quadratic Poincaré gauge theory for general backgrounds

Cubic Poincaré gauge theory

- Cubic parity preserving branch with mixing terms:[3]

$$\mathcal{L}^{(3)}_{\rm curv-tors} = \mathcal{L}^{(3)}_{\tilde{R}TT} + \mathcal{L}^{(3)}_{\tilde{R}SS} + \mathcal{L}^{(3)}_{\tilde{R}tt} + \mathcal{L}^{(3)}_{\tilde{R}TS} + \mathcal{L}^{(3)}_{\tilde{R}Tt} + \mathcal{L}^{(3)}_{\tilde{R}St}\,,$$

$$\begin{aligned}
\mathcal{L}^{(3)}_{\tilde{R}TT} &= h_1 \tilde{R}_{\mu\nu} T^{\mu} T^{\nu} + h_2 \tilde{R} T_{\mu} T^{\mu}\,, \quad \mathcal{L}^{(3)}_{\tilde{R}SS} = h_3 \tilde{R}_{\mu\nu} S^{\mu} S^{\nu} + h_4 \tilde{R} S_{\mu} S^{\mu}\,,\\
\mathcal{L}^{(3)}_{\tilde{R}tt} &= h_5 \tilde{R}_{\lambda\rho\mu\nu} t_{\sigma}{}^{\lambda\rho} t^{\sigma\mu\nu} + h_6 \tilde{R}_{\lambda\rho\mu\nu} t_{\sigma}{}^{\lambda\mu} t^{\sigma\rho\nu} + h_7 \tilde{R}_{\lambda\rho\mu\nu} t^{\lambda\rho}{}_{\sigma} t^{\sigma\mu\nu}\\
&+ h_8 \tilde{R}_{\lambda\rho\mu\nu} t^{\lambda\mu}{}_{\sigma} t^{\sigma\rho\nu} + h_9 \tilde{R}_{\lambda\rho\mu\nu} t^{\lambda\mu}{}_{\sigma} t^{\rho\nu\sigma} + h_{10} \tilde{R}_{\lambda\rho} t_{\mu\nu}{}^{\lambda} t^{\rho\mu\nu}\\
&+ h_{11} \tilde{R}_{\lambda\rho} t_{\mu\nu}{}^{\lambda} t^{\mu\nu\rho} + h_{12} \tilde{R} t_{\lambda\rho\mu} t^{\lambda\rho\mu}\,,\\
\mathcal{L}^{(3)}_{\tilde{R}TS} &= h_{13} \varepsilon^{\lambda\rho\mu\nu} \tilde{R}_{\lambda\rho\mu\nu} T_{\sigma} S^{\sigma} + h_{14} \varepsilon_{\nu}{}^{\lambda\rho\sigma} \tilde{R}_{\lambda\rho\mu\sigma} T^{\mu} S^{\nu} + h_{15} \varepsilon^{\lambda\rho\mu\nu} \tilde{R}_{\lambda\rho} T_{\mu} S_{\nu}\,,\\
\mathcal{L}^{(3)}_{\tilde{R}Tt} &= h_{16} \tilde{R}_{\lambda\rho\mu\nu} T^{\nu} t^{\lambda\rho\mu} + h_{17} \tilde{R}_{\lambda\rho\mu\nu} T^{\rho} t^{\lambda\mu\nu} + h_{18} \tilde{R}_{\lambda\rho} T_{\mu} t^{\mu\lambda\rho} + h_{19} \tilde{R}_{\lambda\rho} T_{\mu} t^{\lambda\rho\mu},\\
\mathcal{L}^{(3)}_{\tilde{R}St} &= h_{20} \varepsilon_{\alpha\rho\mu\nu} \tilde{R}_{\tau}{}^{\rho\mu\nu} S^{\gamma} t^{\alpha\tau}{}_{\gamma} + h_{21} \varepsilon_{\alpha\rho\mu\nu} \tilde{R}_{\tau}{}^{\rho\mu\nu} S^{\gamma} t_{\gamma}{}^{\alpha\tau}\\
&+ h_{22} \varepsilon_{\alpha\rho}{}^{\mu\nu} \tilde{R}^{\rho}{}_{\mu\tau\nu} S^{\gamma} t_{\gamma}{}^{\alpha\tau} + h_{23} \varepsilon_{\alpha\rho}{}^{\mu\nu} \tilde{R}_{\gamma\mu\tau\nu} S^{\alpha} t^{\gamma\rho\tau}\\
&+ h_{24} \varepsilon_{\alpha\rho}{}^{\mu\nu} \tilde{R}_{\gamma\mu\tau\nu} S^{\alpha} t^{\rho\tau\gamma} + h_{25} \varepsilon_{\alpha\rho\tau\mu} \tilde{R}^{\mu}{}_{\gamma} S^{\alpha} t^{\rho\tau\gamma} + h_{26} \varepsilon_{\lambda\rho\mu\nu} \tilde{R}^{\lambda\rho} S_{\sigma} t^{\sigma\mu\nu}\,.
\end{aligned}$$

[3] S. Bahamonde and J. Gigante Valcarcel, Phys. Rev. D **109** (2024) no.10, 10

- Stable vector and axial sectors:

$$
\begin{aligned}
\mathcal{L} \propto\ & -R + \frac{1}{9}\left(4c_1 + c_2 + 2d_1\right) F^{(T)}_{\mu\nu} F^{(T)\mu\nu} - \frac{1}{72}\left(4c_1 + c_2 + d_1\right) F^{(S)}_{\mu\nu} F^{(S)\mu\nu} \\
& + \frac{1}{24}\left(c_2 - 2c_1\right) \nabla_\mu S^\mu \nabla_\nu S^\nu + \frac{1}{54}\left[2c_1 - 4c_2 - 9\left(4h_3 + 6h_{13} + h_{14}\right)\right] S^\mu S^\nu \nabla_\mu T_\nu \\
& - \frac{1}{108}\left[4c_1 + c_2 + 9\left(4h_3 + 24h_4 + 2h_{14} - h_{15}\right)\right] S_\mu S^\mu \nabla_\nu T^\nu \\
& + \frac{1}{18}\left[2c_1 - c_2 - 3\left(6h_{13} + h_{14} - h_{15}\right)\right] T^\mu S^\nu \nabla_\nu S_\mu - 2h_2 T_\mu T^\mu \nabla_\nu T^\nu \\
& - \frac{2}{3}\left(h_{14} - h_{15}\right) \varepsilon^{\lambda\rho\mu\nu} T_\lambda S_\rho \partial_\mu T_\nu + \frac{1}{36}\left(4c_1 + c_2 + 36h_3\right) G_{\mu\nu} S^\mu S^\nu \\
& + \frac{1}{72}\left[4c_1 + c_2 + 36\left(h_3 + 2h_4\right)\right] R S_\mu S^\mu + h_1 G_{\mu\nu} T^\mu T^\nu + \frac{1}{2}\left(h_1 + 2h_2\right) R T_\mu T^\mu \\
& - \frac{1}{648}\left[16c_1 + 4c_2 - 9\left(h_1 + 3h_2 - 16h_3 - 48h_4 - 8h_{14}\right)\right] T_\mu T^\mu S_\nu S^\nu \\
& + \frac{1}{648}\left[16c_2 - 8c_1 - 9\left(h_1 - 16h_3 - 48h_{13} - 8h_{14}\right)\right] T_\mu S^\mu T_\nu S^\nu \\
& - \frac{2}{3} h_2 T_\mu T^\mu T_\nu T^\nu + \frac{1}{24} h_4 S_\mu S^\mu S_\nu S^\nu + \frac{1}{2} m_T^2 T_\mu T^\mu + \frac{1}{2} m_S^2 S_\mu S^\mu \,.
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- First stability constraints (6 equalities):

$$c_2 = 2c_1\,, \quad h_2 = -\,\frac{h_1}{2}\,, \quad h_3 = -\,\frac{1}{6}\left(c_1 + 6h_{13}\right)\,,$$
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Vector and axial sectors in cubic Poincaré gauge theory

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- Reduced Lagrangian:

$$\begin{aligned}\mathcal{L} \propto\; & -R + \frac{2}{9}\left(3c_1 + d_1\right) F^{(T)}_{\mu\nu} F^{(T)\mu\nu} - \frac{1}{72}\left(6c_1 + d_1\right) F^{(S)}_{\mu\nu} F^{(S)\mu\nu} \\ & + 2h_1 T_\mu T^\mu \nabla_\nu T^\nu + 4h_{13}\varepsilon^{\lambda\rho\mu\nu} T_\lambda S_\rho \partial_\mu T_\nu + h_1 G_{\mu\nu} T^\mu T^\nu - h_{13} G_{\mu\nu} S^\mu S^\nu \\ & + \frac{1}{144}\left(16h_{13} - h_1\right) T_\mu T^\mu S_\nu S^\nu + \frac{1}{72}\left(16h_{13} - h_1\right) T_\mu S^\mu T_\nu S^\nu \\ & + \frac{1}{3} h_1 T_\mu T^\mu T_\nu T^\nu + \frac{1}{48} h_{13} S_\mu S^\mu S_\nu S^\nu + \frac{1}{2} m_T^2 T_\mu T^\mu + \frac{1}{2} m_S^2 S_\mu S^\mu\,.\end{aligned}$$

Complete theory contains 23+3 constants. Clearly $-\frac{d_1}{6} \leq c_1 \leq -\frac{d_1}{3}\,, d_1 \leq 0$.

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Problem solved

By adding Cubic interactions, it is possible to construct a stable theory with propagating torsion in the gauge formalism

Quadratic Metric-Affine Gravity - Vector/Axial Stability

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Problem solved in MAG as well

By following a similar and much more complex analysis: "By adding Cubic interactions, it is possible to construct a stable theory with propagating torsion and nonmetricity in the gauge formalism"

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New exact black hole solution with three intrinsic charges

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- It is important to mention that the in this theory both torsion and nonmetricity are propagating and moreover, their spin-2 and spin-3 parts are dynamical.
- All the masses of the tensor modes of torsion and nonmetricity are different from zero $\Longrightarrow$ We evaded the Weinberg-Witten no-go theorem (massless higher-spin fields are pathological)

Overview of the Talk

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- Teleparallel theories are restricted geometries where curvature vanishes. One can formulate the same theory as GR with torsion or nonmetricity and then modify them

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- The general MAG is a gauge theory of gravity where torsion, nonmetricity and curvature appear as field strength tensors.
- Within MAG, we constructed the cosmological perturbations where torsion/nonmetricity can be propagating.

Conclusions

- The well-known instabilities in the vector and axial sectors of quadratic Poincaré gauge theory (torsion cannot be propagating) and also Metric-Affine gravity can be eliminated by introducing cubic order invariants in the gravitational action.

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- For the first time we found a solution with all the intrinsic charges without known instabilities and with all higher-spin masses of torsion and nonmetricity.
- Due to the difficulty of these theories, it is important to understand certain symmetries or special important cases for the fields. To understand this, we formulated the algebraic classification of MAG for the curvature tensor. This is analogous to the Petrov classification.

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- Study the axial symmetry of the Poincaré case to find a gravitational spin-orbit interaction (interaction between the intrinsic spin and the angular momentum)
- It is possible to find a renormalizable theory with these cubic interactions?