

Metric-Affine theories of gravity and their applications

Sebastián Bahamonde

Postdoctoral Researcher at Kavli IPMU, University of Tokyo, Japan

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東京大学
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- 2 Teleparallel theories
 - Trinity of Gravity
 - Modified Theories with torsion and applications (Metric TG)
 - Theories with Nonmetricity and applications (Symmetric TG)
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- 5 Stability in Metric Affine Gauge theories
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- In particular, it measures the change of vector components on parallel transport along an infinitesimal closed curve:

$$\delta V_{\mu} = \tilde{R}^{\lambda}{}_{\mu\rho\nu} V_{\lambda} ds^{\rho\nu} ,$$

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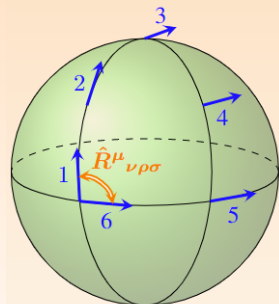
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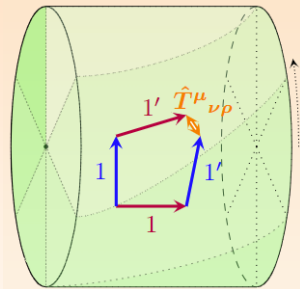
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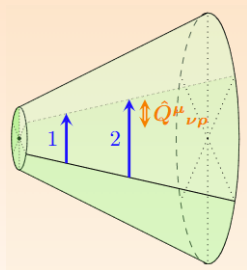
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Post-Riemannian decomposition

- According to the Fundamental Theorem of Riemannian geometry, in the absence of torsion and nonmetricity:

$$\begin{aligned}\tilde{\Gamma}^{\lambda}{}_{\mu\nu} &= \Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} (\partial_{\mu} g_{\nu\rho} + \partial_{\nu} g_{\mu\rho} - \partial_{\rho} g_{\mu\nu}) , \\ \tilde{R}_{\lambda\rho\mu\nu} &= R_{\lambda\rho\mu\nu} = \frac{1}{2} \left(\frac{\partial^2 g_{\lambda\nu}}{\partial x^{\rho} \partial x^{\mu}} + \frac{\partial^2 g_{\rho\mu}}{\partial x^{\lambda} \partial x^{\nu}} - \frac{\partial^2 g_{\lambda\mu}}{\partial x^{\rho} \partial x^{\nu}} - \frac{\partial^2 g_{\rho\nu}}{\partial x^{\lambda} \partial x^{\mu}} \right) \\ &\quad + g_{\sigma\omega} (\Gamma^{\omega}{}_{\rho\mu} \Gamma^{\sigma}{}_{\lambda\nu} - \Gamma^{\omega}{}_{\rho\nu} \Gamma^{\sigma}{}_{\lambda\mu}) .\end{aligned}$$

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- General decomposition of the affine connection and the curvature tensor:

$$\begin{aligned}\tilde{\Gamma}^{\lambda}{}_{\mu\nu} &= \Gamma^{\lambda}{}_{\mu\nu} + N^{\lambda}{}_{\mu\nu} , \\ N^{\lambda}{}_{\mu\nu} &= \frac{1}{2} T^{\lambda}{}_{\mu\nu} - T_{(\mu}{}^{\lambda}{}_{\nu)} + \frac{1}{2} Q^{\lambda}{}_{\mu\nu} - Q_{(\mu}{}^{\lambda}{}_{\nu)} , \\ \tilde{R}^{\lambda}{}_{\rho\mu\nu} &= R^{\lambda}{}_{\rho\mu\nu} + 2\nabla_{[\mu} N^{\lambda}{}_{\rho|\nu]} + 2N^{\lambda}{}_{\sigma[\mu} N^{\sigma}{}_{\rho|\nu]} .\end{aligned}$$

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- Three independent rank-2 tensors defined from the first contractions of the curvature tensor:

$$\tilde{R}_{\mu\nu} = \tilde{R}^{\lambda}{}_{\mu\lambda\nu}, \quad \hat{R}_{\mu\nu} = \tilde{R}_{\mu}{}^{\lambda}{}_{\nu\lambda}, \quad \tilde{R}^{\lambda}{}_{\lambda\mu\nu} = \nabla_{[\nu} Q_{\mu]\lambda}{}^{\lambda}.$$

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- Unique scalar and pseudoscalar curvatures:

$$\tilde{R} = g^{\mu\nu} \tilde{R}_{\mu\nu}, \quad *\tilde{R} = \varepsilon^{\lambda\rho\mu\nu} \tilde{R}_{\lambda\rho\mu\nu}.$$

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Ricci scalar decomposition

$$\tilde{R} = R + \left(T + 2\mathring{\nabla}_\mu(\sqrt{-g}T^\rho{}_\rho{}^\mu) \right) + \left(Q + \mathring{\nabla}_\mu Q^{\mu\nu}{}_\nu - \mathring{\nabla}_\nu Q_\mu{}^{\mu\nu} \right) + C = 0$$

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with

$$T := T^{\rho\lambda\kappa}T_{\rho\lambda\kappa} + 2T^{\rho\lambda\kappa}T_{\kappa\rho\lambda} - 4T_\rho{}^\kappa{}_\kappa T^{\rho\lambda}{}_\lambda, \quad \text{Torsion scalar,}$$

$$Q := -\frac{1}{4} Q_{\alpha\beta\gamma} Q^{\alpha\beta\gamma} + \frac{1}{2} Q_{\alpha\beta\gamma} Q^{\beta\alpha\gamma} + \frac{1}{4} Q_\alpha Q^\alpha - \frac{1}{2} Q_\alpha \bar{Q}^\alpha, \quad \text{Nonmetricity scalar,}$$

$$C := 2(Q_{\kappa\rho\lambda} T^{\lambda\kappa\rho} + Q_\rho{}^\sigma T^{\rho\kappa}{}_\kappa - Q^\sigma{}_{\sigma\rho} T^{\rho\kappa}{}_\kappa).$$

- Torsional teleparallel theories (or Metric teleparallelism) assumes $Q = \tilde{R} = 0$ and then $R = -T + B$ and then the TEGR:

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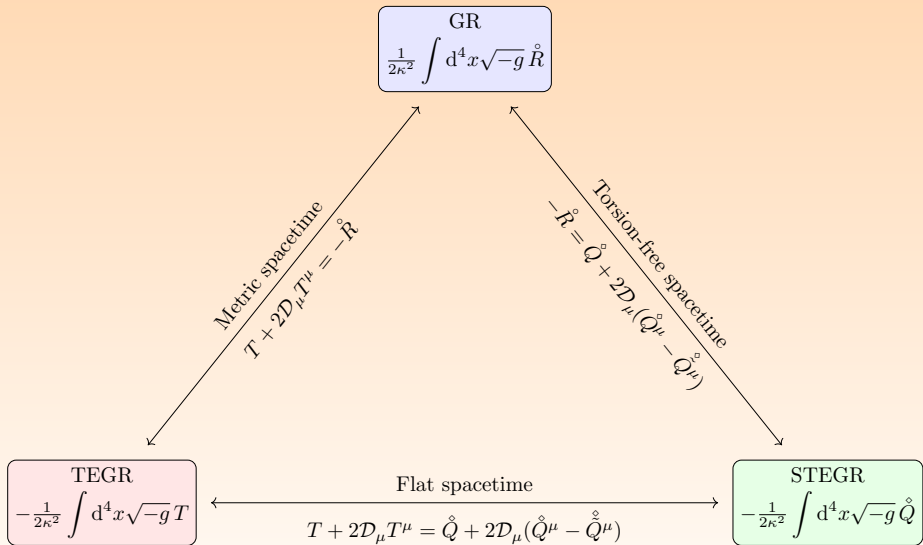


Figure: Geometrical trinity of gravity (S. Bahamonde et.al., “Teleparallel Gravity: From Theory to Cosmology,” Rept. Prog. Phys. **86** (2023) no.2, 026901.; J. Beltrán Jiménez, L. Heisenberg and T. S. Koivisto, “The Geometrical Trinity of Gravity,” Universe **5** (2019) no.7, 173.)

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 - ① 1-parameter NGR - but it is strongly coupled J. Beltrán Jiménez and K. F. Dialektopoulos, JCAP **01** (2020), 018
 - ② Type 3 NGR - new branch healthy just found S. Bahamonde, D. Blixt, K. F. Dialektopoulos and A. Hell, arXiv:2404.02972.

- Another popular modification is R. Ferraro and F. Fiorini, Phys. Rev. D **75** (2007), 084031

$$\mathcal{L}_{f(T)} = f(T).$$

- ① It has been highly analysed in cosmology - it gives interesting phenomenology.
- ② However, it is strongly coupled in Minkowski and even in non-flat FLRW S. Bahamonde, K. F. Dialektopoulos, M. Hohmann, J. Levi Said, C. Pfeifer and E. N. Saridakis, Eur. Phys. J. C **83** (2023) no.3, 193

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- More theories to analyse such as Teleparallel Horndeksi where $G_5 \neq 0$ is avoided! S. Bahamonde, K. F. Dialektopoulos and J. Levi Said, Phys. Rev. D **100** (2019) no.6, 064018; S. Bahamonde,

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- This theory is a subset of Symmetric Teleparallel Horndeski S. Bahamonde, G. Trenkler, L. G. Trombetta and M. Yamaguchi, Phys. Rev. D **107** (2023) no.10, 104024.

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Gauge formalism of metric-affine geometry

- $A(4, \mathbb{R}) = T^4 \rtimes GL(4, \mathbb{R})$ gauge connection with an independent local metric structure¹:

$$A_\mu = e^a{}_\mu P_a + \omega^a{}_{b\mu} L_a{}^b,$$

$$g_{\mu\nu} = e^a{}_\mu e^b{}_\nu g_{ab},$$

where

$$\omega^a{}_{b\mu} = e^a{}_\lambda e_b{}^\rho \tilde{\Gamma}^\lambda{}_{\rho\mu} + e^a{}_\lambda \partial_\mu e_b{}^\lambda.$$

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- Generators of the affine group $A(4, \mathbb{R})$:

$$[P_a, P_b] = 0,$$
$$[L_a{}^b, P_c] = i \delta^b{}_c P_a,$$
$$[L_a{}^b, L_c{}^d] = i(\delta^b{}_c L_a{}^d - \delta_a{}^d L_c{}^b).$$

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$$F_{\mu\nu} = F^a{}_{\mu\nu} P_a + F^a{}_{b\mu\nu} J_a{}^b,$$

where

$$\begin{aligned} F^a{}_{\mu\nu} &= \partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu + \omega^a{}_{b\mu} e^b{}_\nu - \omega^a{}_{b\nu} e^b{}_\mu, \\ F^a{}_{b\mu\nu} &= \partial_\mu \omega^a{}_{b\nu} - \partial_\nu \omega^a{}_{b\mu} + \omega^a{}_{c\mu} \omega^c{}_{b\nu} - \omega^a{}_{c\nu} \omega^c{}_{b\mu}. \end{aligned}$$

Poincaré gauge theory

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- Correspondence with the torsion and curvature tensors:

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- Correspondence with the curvature, torsion and nonmetricity tensors:

$$\begin{aligned}G_{ab\mu} &= g_{ac} g_{bd} e^{c\lambda} e^{d\rho} Q_{\mu\lambda\rho}, \\F^a{}_{\mu\nu} &= e^a{}_\lambda T^\lambda{}_{\nu\mu}, \\F^a{}_{b\mu\nu} &= g_{bc} e^a{}_\lambda e^{c\rho} \tilde{R}^\lambda{}_{\rho\mu\nu}.\end{aligned}$$

- Gravitational action with dynamical torsion and nonmetricity

$$S = \int d^4x \sqrt{-g} \left[\mathcal{L}_m - \frac{1}{16\pi} \mathcal{L}_g(\tilde{\mathcal{R}}, \mathcal{T}, \mathcal{Q}) \right].$$

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$$\frac{1}{\sqrt{-g}} \frac{\delta (\mathcal{L}_g \sqrt{-g})}{\delta e^a{}_\nu} = 16\pi \theta_a{}^\nu,$$
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3 + 1 decomposition - settling notation

- Let us first introduce a projector tensor

$$P_{\mu\nu} \equiv g_{\mu\nu} + n_{\mu}n_{\nu} ,$$

which is orthogonal ($P_{\mu\nu}n^{\mu} = 0$) to a timelike vector n_{μ} satisfying $n_{\mu}n^{\mu} = -1$.

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- Let us introduce arrows on top of quantities to denote traceless-spatial and symmetric quantities, i.e.:

$$\vec{X}_{\mu_1 \dots \mu_n} \equiv P_{\mu_1}^{\alpha_1} \dots P_{\mu_n}^{\alpha_n} X_{\alpha_1 \dots \alpha_n},$$

$$\vec{X}_{\mu_1 \dots \mu_n} = \vec{X}_{(\mu_1 \dots \mu_n)},$$

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- One can then straightforwardly decompose scalar and vectors:

$$\phi = \phi, \quad X_\mu = \vec{X}_\mu - n_\mu X_*.$$

Cosmology in MAG

- Let us assume that the metric, torsion and nonmetricity have the same cosmological symmetries (isotropy and homogeneity)

$$\mathcal{L}_\xi \bar{g}_{\mu\nu} = \mathcal{L}_\xi \bar{T}^\lambda{}_{\mu\nu} = \mathcal{L}_\xi \bar{Q}^\lambda{}_{\mu\nu} = 0.$$

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- By solving those equations one get the FLRW metric and torsion+nonmetricity satisfying homogeneity and isotropy:

$$\begin{aligned}\bar{g} &= -\bar{n}_\mu \bar{n}_\nu dx^\mu \otimes dx^\nu + \bar{P}_{\mu\nu} dx^\mu \otimes dx^\nu = -N^2 dt \otimes dt + a^2 \gamma_{ij} dx^i \otimes dx^j, \\ \bar{T}^\lambda{}_{\mu\nu} &= 2T_1(t) \bar{n}_{[\mu} \bar{P}_{\nu]}{}^\lambda + 2T_2(t) \bar{\varepsilon}^\lambda{}_{\mu\nu\rho} \bar{n}^\rho, \\ \bar{Q}^\lambda{}_{\mu\nu} &= 2Q_1(t) \bar{n}_\lambda \bar{n}_\mu \bar{n}_\nu + 2Q_2(t) \bar{n}_\lambda \bar{P}_{\mu\nu} + 2Q_3(t) \bar{P}_{\lambda(\mu} \bar{n}_{\nu)},\end{aligned}$$

where $\gamma_{ij} dx^i \otimes dx^j = \frac{dr^2}{1-Kr^2} + r^2 d\Omega^2$. Note that there are 5 independent functions coming from Post-Riemannian.

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- Our aim is to perform a 3 + 1+SVT-decomposition for all the tensors in MAG up to linear perturbations to then write:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}, \quad \bar{T}^\lambda{}_{\mu\nu} = \bar{T}^\lambda{}_{\mu\nu} + \delta T^\lambda{}_{\mu\nu}, \quad \bar{Q}^\lambda{}_{\mu\nu} = \bar{Q}^\lambda{}_{\mu\nu} + \delta Q^\lambda{}_{\mu\nu}.$$

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- Any realistic theory should describe both background and cosmological perturbations observables (for example CMB, dark energy, large scale structure, etc.)
- So far, the majority of the studies in MAG have been concentrated on describing background cosmology only. A few models (very specific) goes beyond background.
- The majority of these studies predominantly focus on torsion.
- Usually, it is possible to avoid cosmological singularities and replace them a cosmic bounce.
- Dark energy can be explained by the scalar modes of torsion.
- It is possible to find inflationary models such as Einstein-Cartan couple to Higgs or other more complicated ones which are compatible with observations.

Decomposition into irreducible parts

Irreducible decomposition of the torsion tensor

$$T^\lambda{}_{\mu\nu} = \frac{1}{3} (\delta^\lambda{}_\nu T_\mu - \delta^\lambda{}_\mu T_\nu) + \frac{1}{6} \varepsilon^\lambda{}_{\rho\mu\nu} S^\rho + t^\lambda{}_{\mu\nu} .$$

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Irreducible decomposition of the nonmetricity tensor

$$Q_{\lambda\mu\nu} = g_{\mu\nu} W_\lambda + \mathcal{Q}_{\lambda\mu\nu},$$

$$\mathcal{Q}_{\lambda\mu\nu} = \frac{1}{2} (g_{\lambda\mu} \Lambda_\nu + g_{\lambda\nu} \Lambda_\mu) - \frac{1}{4} g_{\mu\nu} \Lambda_\lambda + \frac{1}{3} \varepsilon_{\lambda\rho\sigma(\mu} \Omega_{\nu)}{}^{\rho\sigma} + q_{\lambda\mu\nu}.$$

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- Second tensor part $q_{\lambda\mu\nu} = Q_{(\lambda\mu\nu)} - g_{(\mu\nu} W_{\lambda)} - \frac{3}{4} g_{(\mu\nu} \Lambda_{\lambda)}$.

3 + 1 decomposition for FLRW - Torsion sector

- The irreducible modes of antisymmetric rank-3 tensor (as torsion) can be decomposed as

$$T_\mu = \vec{T}_\mu - n_\mu \phi,$$

$$S_\mu = \vec{S}_\mu - n_\mu \varrho,$$

$$t_{\mu\nu\rho} = 2n_{[\nu}\vec{A}_{\rho]\mu} + 2\left(n_\mu n_{[\nu} + \frac{1}{2}P_{\mu[\nu}\right)\vec{B}_{\rho]} + \frac{1}{2}\varepsilon_{\nu\rho}{}^{\alpha\beta}\left[-n_\alpha\vec{A}_{\beta\mu} + \left(n_\beta n_\mu + \frac{1}{2}P_{\beta\mu}\right)\vec{B}_\alpha\right].$$

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- Then, torsion is decomposed as

$$T^\lambda{}_{\mu\nu} = \bar{T}^\lambda{}_{\mu\nu} + \delta T^\lambda{}_{\mu\nu} = \begin{cases} T^0{}_{0i} = -\frac{a}{3}(\vec{T}_i + 3\vec{B}_i) \\ T^0{}_{ij} = -\frac{a^2}{6N}\varepsilon_{ijk}(\vec{S}^k + 3\vec{B}^k) \\ T^i{}_{0j} = -N\left[\vec{A}^i{}_j - \frac{1}{3}\delta^i{}_j(\bar{\phi} + \phi) + \frac{1}{6}\varepsilon^i{}_{jk}(\vec{S}^k - \frac{3}{2}\vec{B}^k)\right] \\ T^i{}_{jk} = -\frac{a}{6}\left\{4\delta^i{}_{[j}\vec{T}_{k]} - 6\delta^i{}_{[j}\vec{B}_{k]} + \left[3\varepsilon_{jkl}\vec{A}^{il} - \varepsilon^i{}_{jk}(\bar{\varrho} + \varrho)\right]\right\} \end{cases}$$

where we have introduced the background T_i in the scalars ϕ, ϱ .

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$$W_\mu = \vec{W}_\mu - n_\mu \theta,$$

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$$q_{\mu\nu\rho} = \vec{C}_{\mu\nu\rho} - 3n_{(\mu} \vec{\kappa}_{\nu\rho)} + \frac{3}{5}\left(5n_{(\mu} n_\nu + P_{(\mu\nu)}\right) \vec{Z}_{\rho)} - \left(n_{(\mu} n_\nu + P_{(\mu\nu)}\right) n_{\rho)} \xi.$$

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- Then, nonmetricity is decomposed as

$$Q^\lambda{}_{\mu\nu} = \begin{cases} Q_{000} = -N^3 \left[\bar{\theta} + \theta + \frac{3}{4}(\bar{\psi} + \psi) - (\bar{\xi} + \xi) \right] \\ Q_{00i} = -\frac{N^2 a}{2} \left(\vec{\Lambda}_i - \frac{1}{3} \vec{Y}_i - 2\vec{Z}_i \right) \\ Q_{0ij} = -\frac{Na^2}{3} \left\{ \vec{Q}_{ij} - 3\vec{\kappa}_{ij} - 3 \left[\bar{\theta} + \theta - \frac{1}{4}(\bar{\psi} + \psi) + \frac{1}{3}(\bar{\xi} + \xi) \right] \gamma_{ij} \right\} \\ Q_{i00} = -N^2 a \left(\vec{W}_i - \frac{1}{4} \vec{\Lambda}_i + \frac{1}{3} \vec{Y}_i - \vec{Z}_i \right) \\ Q_{i0j} = \frac{Na^2}{6} \left\{ [\vec{Q}_{ij} + 6\vec{\kappa}_{ij} + (3(\bar{\psi} + \psi) + 2(\bar{\xi} + \xi)) \gamma_{ij}] - 3\varepsilon_{ijk} \vec{Y}^k \right\} \\ Q_{ijk} = a^3 \left\{ \vec{C}_{ijk} + \frac{1}{4}(4\vec{W}_i - \vec{\Lambda}_i) \gamma_{jk} + \gamma_{i(j} \vec{\Lambda}_{k)} - \frac{1}{6}(\gamma_{jk} \vec{Y}_i - \gamma_{i(j} \vec{Y}_{k)}) + \frac{3}{5} \gamma_{(ij} \vec{Z}_{k)} + \frac{2}{3} \varepsilon_{li(j} \vec{Q}^l{}_{k)} \right\} \end{cases}$$

where we have introduced the background Q_i in the scalars θ, ψ, ξ .

Summary of modes

- We can group the modes as:

$$\delta\vec{\mathbf{X}} = \{\alpha, \zeta, \phi, \varrho, \theta, \psi, \xi\},$$

$$\delta\vec{\mathbf{X}}_i = \{\vec{\beta}_i, \vec{T}_i, \vec{S}_i, \vec{B}_i, \vec{\mathcal{B}}_i, \vec{W}_i, \vec{\Lambda}_i, \vec{Y}_i, \vec{\mathcal{Y}}_i, \vec{Z}_i\},$$

$$\delta\vec{\mathbf{X}}_{ij} = \{\vec{h}_{ij}, \vec{A}_{ij}, \vec{\mathcal{A}}_{ij}, \vec{Q}_{ij}, \vec{\mathcal{Q}}_{ij}, \vec{\kappa}_{ij}\},$$

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Spin and Parity	0 ⁺	0 ⁻	1 ⁺	1 ⁻	2 ⁺	2 ⁻	3 ⁻
Metric sector $g_{\mu\nu}$	α, ζ	-	-	$\vec{\beta}_i$	\vec{h}_{ij}	-	-
Torsion sector $T^\lambda{}_{\mu\nu}$	ϕ	ϱ	$\vec{S}_i, \vec{\mathcal{B}}_i$	\vec{T}_i, \vec{B}_i	\vec{A}_{ij}	$\vec{\mathcal{A}}_{ij}$	-
Nonmetricity sector $Q_{\lambda\mu\nu}$	θ, ψ, ξ	-	$\vec{\mathcal{Y}}_i$	$\vec{W}_i, \vec{\Lambda}_i, \vec{Y}_i, \vec{Z}_i$	$\vec{Q}_{ij}, \vec{\kappa}_{ij}$	$\vec{\mathcal{Q}}_{ij}$	\vec{C}_{ijk}

Table: Species in MAG. K. Aoki, S. Bahamonde, J. Gigante Valcarcel and M. A. Gorji, "Cosmological Perturbation Theory in Metric-Affine Gravity," [arXiv:2310.16007 [gr-qc]].

Helicity (SVT) decomposition

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- After the 3 + 1 decomposition, we only need to deal with the spatial scalar and tensors with Latin indices belonging to the maximally symmetric space γ_{ij} (and with the “external” parameter t).
- Let us decompose them into different helicity sectors:

$$\delta \vec{\mathbf{X}}_i = D_i \mathbf{S} + \mathbf{V}_i^{(T)},$$

$$\delta \vec{\mathbf{X}}_{ij} = \left(D_{(i} D_{j)} - \frac{1}{3} \gamma_{ij} D^2 \right) \mathbf{S} + D_{(i} \mathbf{V}_{j)}^{(T)} + \mathbf{T}_{ij}^{(TT)},$$

$$\delta \vec{\mathbf{X}}_{ijk} = \left[D_{(i} D_j D_{k)} - \frac{1}{5} \gamma_{(ij} D_{k)} (3D^2 + 4K) \right] \mathbf{S} + \left[D_{(i} D_{j)} - \frac{1}{5} \gamma_{(ij} (D^2 + 2K) \right] \mathbf{V}_{k)}^{(T)} + D_{(i} \mathbf{T}_{jk)}^{(TT)}$$

Here, the superscript “(T)” refers that the quantity is transverse, while the tensors denoted by the superscript “(TT)” are transverse-traceless and symmetric:

$$D^i \mathbf{V}_i^{(T)} = 0,$$

$$D^i \mathbf{T}_{ij}^{(TT)} = \mathbf{T}_{ij}^{(TT)} \gamma^{ij} = 0, \quad \mathbf{T}_{ij}^{(TT)} = \mathbf{T}_{(ij)}^{(TT)},$$

$$D^i \mathbf{T}_{ijk}^{(TT)} = \mathbf{T}_{ijk}^{(TT)} \gamma^{ij} = 0, \quad \mathbf{T}_{ijk}^{(TT)} = \mathbf{T}_{(ijk)}^{(TT)}.$$

SVT decomposition around FRW

- Thereby, the 10 dof described by the metric perturbations are split in terms of four scalars $\{\alpha, \beta, \zeta, h\}$ (1 dof each), two transverse vectors $\{\beta_i^{(T)}, h_i^{(T)}\}$ (2 dof each), and one symmetric and transverse-traceless tensor $h_{ij}^{(TT)}$ (2 dof).

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SVT	Quantities	dof	Total dof
5 scalars	$\{T, B, \phi, A, \varrho\}$	1 dof each	5
3 pseudoscalars	$\{\mathcal{S}, \mathcal{B}, \mathcal{A}\}$	1 dof each	3
3 vectors	$\{T_i^{(T)}, B_i^{(T)}, A_i^{(T)}\}$	2 dof each	6
3 pseudovectors	$\{\mathcal{S}_i^{(T)}, \mathcal{B}_i^{(T)}, \mathcal{A}_i^{(T)}\}$	2 dof each	6
1 rank-2 tensor	$\{A_{ij}^{(TT)}\}$	2 dof each	2
1 rank-2 pseudotensor	$\{\mathcal{A}_{ij}^{(TT)}\}$	2 dof each	2

Table: Perturbation spectrum for the torsion tensor. K. Aoki, S. Bahamonde, J. Gigante Valcarcel and M. A. Gorji, "Cosmological Perturbation Theory in Metric-Affine Gravity," [arXiv:2310.16007 [gr-qc]].

SVT decomposition around FRW

SVT	Quantities	dof	Total dof
10 scalars	$\{\theta, \psi, \xi, \Lambda, Y, Z, \kappa, Q, W, C\}$	1 dof each	10
2 pseudoscalars	$\{\mathcal{Y}, \mathcal{Q}\}$	1 dof each	2
7 vectors	$\{\Lambda^{(T)}_i, Y^{(T)}_i, Z^{(T)}_i, \kappa^{(T)}_i, Q^{(T)}_i, W^{(T)}_i, C^{(T)}_i\}$	2 dof each	14
2 pseudovectors	$\{\mathcal{Y}^{(T)}_i, \mathcal{Q}^{(T)}_i\}$	2 dof each	4
3 rank-2 tensor	$\{\kappa^{(TT)}_{ij}, Q^{(TT)}_{ij}, C^{(TT)}_{ij}\}$	2 dof each	6
1 rank-2 pseudotensor	$\{\mathcal{Q}^{(TT)}_{ij}\}$	2 dof each	2
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Table: Perturbation spectrum for the nonmetricity tensor.

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Table: Perturbation spectrum for the nonmetricity tensor.

- Note that the spin-3 carries a helicity-3 part and since the metric does not have such terms, it totally decouples from the other modes in FLRW.

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Table: Perturbation spectrum for the nonmetricity tensor.

- Note that the spin-3 carries a helicity-3 part and since the metric does not have such terms, it totally decouples from the other modes in FLRW.
- We studied this spin-3 part finding

$$\ddot{C}^{(A)} + 3H\dot{C}^{(A)} + [\omega_{C,k}^{(A)}]^2 C^{(A)} = J^{(A)}.$$

Overview of the Talk

- 1 Motivations and introduction to metric-affine geometry
- 2 Teleparallel theories
 - Trinity of Gravity
 - Modified Theories with torsion and applications (Metric TG)
 - Theories with Nonmetricity and applications (Symmetric TG)
- 3 Gauge theories of gravity
- 4 Cosmological perturbations in MAG
- 5 Stability in Metric Affine Gauge theories**
 - **Stability issues in quadratic Poincaré gauge theory**
 - **Cubic extensions**
- 6 Algebraic Classification in MAG
- 7 Conclusions and future prospects

- Class of quadratic Poincaré gauge models that are reduced to General Relativity in the absence of torsion:

$$S_g = \frac{1}{16\pi} \int \left[c_2 \tilde{R}_{\lambda\rho\mu\nu} \tilde{R}^{\lambda\mu\rho\nu} - R - \frac{1}{2} (2c_1 + c_2) \tilde{R}_{\lambda\rho\mu\nu} \tilde{R}^{\mu\nu\lambda\rho} + c_1 \tilde{R}_{\lambda\rho\mu\nu} \tilde{R}^{\lambda\rho\mu\nu} + d_1 \tilde{R}_{\mu\nu} (\tilde{R}^{\mu\nu} - \tilde{R}^{\nu\mu}) + \frac{1}{2} (m_T^2 T_\mu T^\mu + m_S^2 S_\mu S^\mu + m_t^2 t_{\lambda\mu\nu} t^{\lambda\mu\nu}) \right] \sqrt{-g} d^4x.$$

- Class of quadratic Poincaré gauge models that are reduced to General Relativity in the absence of torsion:

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- Focusing on the vector and axial modes of torsion:

$$\begin{aligned} \tilde{R}_{\lambda\rho\mu\nu} &= R_{\lambda\rho\mu\nu} + \frac{2}{3} (g_{\lambda[\nu} \nabla_{\mu]} T_{\rho} - g_{\rho[\nu} \nabla_{\mu]} T_{\lambda}) + \frac{1}{6} \varepsilon_{\lambda\rho\sigma[\mu} \nabla_{\nu]} S^\sigma \\ &+ \frac{2}{9} (2T_{[\lambda} g_{\rho][\nu} T_{\mu]} + g_{\lambda[\nu} g_{\mu]\rho} T_\sigma T^\sigma) + \frac{1}{72} (2S_{[\lambda} g_{\rho][\mu} S_{\nu]} + g_{\lambda[\mu} g_{\nu]\rho} S_\sigma S^\sigma) \\ &+ \frac{1}{18} [2\varepsilon_{\mu\nu\sigma[\lambda} T_{\rho]} S^\sigma + (g_{\lambda[\nu} \varepsilon_{\mu]\rho\sigma\omega} - g_{\rho[\nu} \varepsilon_{\mu]\lambda\sigma\omega}) T^\sigma S^\omega] , \\ \tilde{R}_{\mu\nu} &= R_{\mu\nu} - \frac{1}{3} (2\nabla_\nu T_\mu + g_{\mu\nu} \nabla_\lambda T^\lambda) + \frac{1}{12} \varepsilon_{\lambda\rho\mu\nu} \nabla^\lambda S^\rho \\ &+ \frac{2}{9} (T_\mu T_\nu - g_{\mu\nu} T_\lambda T^\lambda) + \frac{1}{72} (g_{\mu\nu} S_\lambda S^\lambda - S_\mu S_\nu) . \end{aligned}$$

- Instabilities in the vector and axial sectors:

$$\begin{aligned}\mathcal{L}_g^{(2)} \propto & \frac{1}{9} (4c_1 + c_2 + 2d_1) F_{\mu\nu}^{(T)} F^{(T)\mu\nu} - \frac{1}{72} (4c_1 + c_2 + d_1) F_{\mu\nu}^{(S)} F^{(S)\mu\nu} \\ & + \frac{1}{24} (c_2 - 2c_1) \nabla_\mu S^\mu \nabla_\nu S^\nu + \frac{1}{27} (c_1 - 2c_2) S^\mu S^\nu \nabla_\mu T_\nu \\ & - \frac{1}{108} (4c_1 + c_2) S_\mu S^\mu \nabla_\nu T^\nu + \frac{1}{18} (2c_1 - c_2) T^\mu S^\nu \nabla_\nu S_\mu \\ & + \frac{1}{36} (4c_1 + c_2) G_{\mu\nu} S^\mu S^\nu + \frac{1}{72} (4c_1 + c_2) R S_\mu S^\mu \\ & - \frac{1}{162} (4c_1 + c_2) T_\mu T^\mu S_\nu S^\nu + \frac{1}{81} (2c_2 - c_1) T_\mu S^\mu T_\nu S^\nu \\ & + \frac{1}{2} m_T^2 T_\mu T^\mu + \frac{1}{2} m_S^2 S_\mu S^\mu .\end{aligned}$$

and then $c_1 = c_2 = d_1 = 0$ (only the masses survives in general backgrounds!)

Vector and axial sectors in quadratic Poincaré gauge theory

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and then $c_1 = c_2 = d_1 = 0$ (only the masses survives in general backgrounds!)

Important problem!

It is not possible to have a stable propagating torsion tensor in quadratic Poincaré gauge theory for general backgrounds

Cubic Poincaré gauge theory

- Cubic parity preserving branch with mixing terms:²

$$\mathcal{L}_{\text{curv-tors}}^{(3)} = \mathcal{L}_{\tilde{R}TT}^{(3)} + \mathcal{L}_{\tilde{R}SS}^{(3)} + \mathcal{L}_{\tilde{R}tt}^{(3)} + \mathcal{L}_{\tilde{R}TS}^{(3)} + \mathcal{L}_{\tilde{R}Tt}^{(3)} + \mathcal{L}_{\tilde{R}St}^{(3)},$$

$$\mathcal{L}_{\tilde{R}TT}^{(3)} = h_1 \tilde{R}_{\mu\nu} T^\mu T^\nu + h_2 \tilde{R} T_\mu T^\mu, \quad \mathcal{L}_{\tilde{R}SS}^{(3)} = h_3 \tilde{R}_{\mu\nu} S^\mu S^\nu + h_4 \tilde{R} S_\mu S^\mu,$$

$$\begin{aligned} \mathcal{L}_{\tilde{R}tt}^{(3)} &= h_5 \tilde{R}_{\lambda\rho\mu\nu} t_\sigma^{\lambda\rho} t^{\sigma\mu\nu} + h_6 \tilde{R}_{\lambda\rho\mu\nu} t_\sigma^{\lambda\mu} t^{\sigma\rho\nu} + h_7 \tilde{R}_{\lambda\rho\mu\nu} t^{\lambda\rho} t^{\sigma\mu\nu} \\ &+ h_8 \tilde{R}_{\lambda\rho\mu\nu} t^{\lambda\mu} t^{\sigma\rho\nu} + h_9 \tilde{R}_{\lambda\rho\mu\nu} t^{\lambda\mu} t^{\rho\nu\sigma} + h_{10} \tilde{R}_{\lambda\rho} t_{\mu\nu}^{\lambda\rho} t^{\rho\mu\nu} \\ &+ h_{11} \tilde{R}_{\lambda\rho} t_{\mu\nu}^{\lambda\rho} t^{\mu\nu\rho} + h_{12} \tilde{R} t_{\lambda\rho} t^{\lambda\rho\mu}, \end{aligned}$$

$$\mathcal{L}_{\tilde{R}TS}^{(3)} = h_{13} \varepsilon^{\lambda\rho\mu\nu} \tilde{R}_{\lambda\rho\mu\nu} T_\sigma S^\sigma + h_{14} \varepsilon_\nu^{\lambda\rho\sigma} \tilde{R}_{\lambda\rho\mu\sigma} T^\mu S^\nu + h_{15} \varepsilon^{\lambda\rho\mu\nu} \tilde{R}_{\lambda\rho} T_\mu S_\nu,$$

$$\mathcal{L}_{\tilde{R}Tt}^{(3)} = h_{16} \tilde{R}_{\lambda\rho\mu\nu} T^\nu t^{\lambda\rho\mu} + h_{17} \tilde{R}_{\lambda\rho\mu\nu} T^\rho t^{\lambda\mu\nu} + h_{18} \tilde{R}_{\lambda\rho} T_\mu t^{\mu\lambda\rho} + h_{19} \tilde{R}_{\lambda\rho} T_\mu t^{\lambda\rho\mu},$$

$$\begin{aligned} \mathcal{L}_{\tilde{R}St}^{(3)} &= h_{20} \varepsilon_{\alpha\rho\mu\nu} \tilde{R}_\tau^{\rho\mu\nu} S^\gamma t^{\alpha\tau}_\gamma + h_{21} \varepsilon_{\alpha\rho\mu\nu} \tilde{R}_\tau^{\rho\mu\nu} S^\gamma t_\gamma^{\alpha\tau} \\ &+ h_{22} \varepsilon_{\alpha\rho}^{\mu\nu} \tilde{R}^\rho_{\mu\tau\nu} S^\gamma t_\gamma^{\alpha\tau} + h_{23} \varepsilon_{\alpha\rho}^{\mu\nu} \tilde{R}_{\gamma\mu\tau\nu} S^\alpha t^{\gamma\rho\tau} \\ &+ h_{24} \varepsilon_{\alpha\rho}^{\mu\nu} \tilde{R}_{\gamma\mu\tau\nu} S^\alpha t^{\rho\tau\gamma} + h_{25} \varepsilon_{\alpha\rho\tau\mu} \tilde{R}^\mu_\gamma S^\alpha t^{\rho\tau\gamma} + h_{26} \varepsilon_{\lambda\rho\mu\nu} \tilde{R}^{\lambda\rho} S_\sigma t^{\sigma\mu\nu}. \end{aligned}$$

²S. Bahamonde and J. Gigante Valcarcel, Phys. Rev. D **109** (2024) no.10, 10

• Stable vector and axial sectors:

$$\begin{aligned}
 \mathcal{L} \propto & -R + \frac{1}{9} (4c_1 + c_2 + 2d_1) F_{\mu\nu}^{(T)} F^{(T)\mu\nu} - \frac{1}{72} (4c_1 + c_2 + d_1) F_{\mu\nu}^{(S)} F^{(S)\mu\nu} \\
 & + \frac{1}{24} (c_2 - 2c_1) \nabla_\mu S^\mu \nabla_\nu S^\nu + \frac{1}{54} [2c_1 - 4c_2 - 9(4h_3 + 6h_{13} + h_{14})] S^\mu S^\nu \nabla_\mu T_\nu \\
 & - \frac{1}{108} [4c_1 + c_2 + 9(4h_3 + 24h_4 + 2h_{14} - h_{15})] S_\mu S^\mu \nabla_\nu T^\nu \\
 & + \frac{1}{18} [2c_1 - c_2 - 3(6h_{13} + h_{14} - h_{15})] T^\mu S^\nu \nabla_\nu S_\mu - 2h_2 T_\mu T^\mu \nabla_\nu T^\nu \\
 & - \frac{2}{3} (h_{14} - h_{15}) \varepsilon^{\lambda\rho\mu\nu} T_\lambda S_\rho \partial_\mu T_\nu + \frac{1}{36} (4c_1 + c_2 + 36h_3) G_{\mu\nu} S^\mu S^\nu \\
 & + \frac{1}{72} [4c_1 + c_2 + 36(h_3 + 2h_4)] R S_\mu S^\mu + h_1 G_{\mu\nu} T^\mu T^\nu + \frac{1}{2} (h_1 + 2h_2) R T_\mu T^\mu \\
 & - \frac{1}{648} [16c_1 + 4c_2 - 9(h_1 + 3h_2 - 16h_3 - 48h_4 - 8h_{14})] T_\mu T^\mu S_\nu S^\nu \\
 & + \frac{1}{648} [16c_2 - 8c_1 - 9(h_1 - 16h_3 - 48h_{13} - 8h_{14})] T_\mu S^\mu T_\nu S^\nu \\
 & - \frac{2}{3} h_2 T_\mu T^\mu T_\nu T^\nu + \frac{1}{24} h_4 S_\mu S^\mu S_\nu S^\nu + \frac{1}{2} m_T^2 T_\mu T^\mu + \frac{1}{2} m_S^2 S_\mu S^\mu .
 \end{aligned}$$

- First stability constraints (6 equalities):

$$c_2 = 2c_1, \quad h_2 = -\frac{h_1}{2}, \quad h_3 = -\frac{1}{6}(c_1 + 6h_{13}),$$
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Complete theory contains 23+3 constants. Clearly $-\frac{d_1}{6} \leq c_1 \leq -\frac{d_1}{3}, d_1 \leq 0$.

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Conclusion 1: Problem solved

By adding Cubic interactions, it is possible to construct a stable theory with propagating torsion in the gauge formalism

Quadratic Metric-Affine Gravity - Vector/Axial Stability

- Let us focus on the axial and vector sectors. Let us recall the decomposition:

$$T^\lambda{}_{\mu\nu} = \frac{1}{3} \left(\delta^\lambda{}_\nu T_\mu - \delta^\lambda{}_\mu T_\nu \right) + \frac{1}{6} \varepsilon^\lambda{}_{\rho\mu\nu} S^\rho + t^\lambda{}_{\mu\nu} ,$$

$$Q_{\lambda\mu\nu} = g_{\mu\nu} W_\lambda + \frac{1}{2} \left(g_{\lambda\mu} \Lambda_\nu + g_{\lambda\nu} \Lambda_\mu \right) - \frac{1}{4} g_{\mu\nu} \Lambda_\lambda + \frac{1}{3} \varepsilon_{\lambda\rho\sigma(\mu} \Omega_{\nu)}{}^{\rho\sigma} + q_{\lambda\mu\nu} .$$

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New question to address

Is it possible to have both torsion and nonmetricity propagating in a stable theory?

Cubic Metric-Affine Gravity

- Similarly as before, we can introduce cubic interactions and add the quadratic MAG Lagrangian (soon in arXiv)

$$\mathcal{L}_{\text{Cubic}} = \mathcal{L}_{\text{Quad}} + \mathcal{L}_{\text{curv-tors}}^{(3)} + \mathcal{L}_{\text{curv-nonm}}^{(3)} + \mathcal{L}_{\text{curv-tor-nonm}}^{(3)}$$

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- The last contribution $\mathcal{L}_{\text{curv-tor-nonm}}^{(3)}$ comes from mixing terms which depend linearly on the curvature tensor and linearly on both torsion and nonmetricity tensors (e.g. $\Lambda_\mu T_\nu \tilde{R}^{\mu\nu}$).

Cubic Metric-Affine Gravity

- Similarly as before, we can introduce cubic interactions and add the quadratic MAG Lagrangian (soon in arXiv)

$$\mathcal{L}_{\text{Cubic}} = \mathcal{L}_{\text{Quad}} + \mathcal{L}_{\text{curv-tors}}^{(3)} + \mathcal{L}_{\text{curv-nonm}}^{(3)} + \mathcal{L}_{\text{curv-tor-nonm}}^{(3)}$$

- The first contribution $\mathcal{L}_{\text{curv-tors}}^{(3)}$ comes from mixing terms which depend linearly on the curvature tensor and quadratically on the torsion tensor (more terms now).
- The second contribution $\mathcal{L}_{\text{curv-nonm}}^{(3)}$ comes from mixing terms which depend linearly on the curvature tensor and quadratically on the nonmetricity tensor (e.g. $W_\mu W_\nu \tilde{R}^{\mu\nu}$)
- The last contribution $\mathcal{L}_{\text{curv-tor-nonm}}^{(3)}$ comes from mixing terms which depend linearly on the curvature tensor and linearly on both torsion and nonmetricity tensors (e.g. $\Lambda_\mu T_\nu \tilde{R}^{\mu\nu}$).

Conclusion 2: Problem solved in MAG as well

By following a similar and much more complex analysis: “By adding Cubic interactions, it is possible to construct a stable theory with propagating torsion and nonmetricity in the gauge formalism”

Reissner-Nordström-like black holes

- Explicit symmetries on the metric and torsion tensors:

$$\mathcal{L}_\xi g_{\mu\nu} = \mathcal{L}_\xi T^\lambda{}_{\mu\nu} = \mathcal{L}_\xi Q_{\lambda\mu\nu} = 0 \implies \mathcal{L}_\xi \tilde{R}^\lambda{}_{\rho\mu\nu} = 0.$$

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- Static and spherically symmetric space-times:

$$\#10 \rightarrow \#2 \left\{ ds^2 = \Psi_1(r) dt^2 - \frac{dr^2}{\Psi_2(r)} - r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2) ; \right.$$

$$\#24 \rightarrow \#8 \left\{ \begin{array}{lll} T^t{}_{tr} & T^r{}_{tr} & T^\vartheta{}_{t\vartheta} \\ T^\vartheta{}_{r\vartheta} & T^\vartheta{}_{t\varphi} & T^\vartheta{}_{r\varphi} \\ T^t{}_{\vartheta\varphi} & T^r{}_{\vartheta\varphi} & \end{array} \right.$$

$$\#40 \rightarrow \#12 \left\{ \begin{array}{lll} Q_{ttt} & Q_{trr} & Q_{ttr} \\ Q_{t\vartheta\vartheta} & Q_{rtt} & Q_{rrr} \\ Q_{rtr} & Q_{r\vartheta\vartheta} & Q_{\vartheta t\vartheta} \\ Q_{\vartheta r\vartheta} & Q_{\vartheta t\varphi} & Q_{\vartheta r\varphi} \end{array} \right.$$

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New exact black hole solution with three intrinsic charges

$$\Psi(r) = 1 - \frac{2m}{r} + \frac{1}{r^2} \left(H_1 \kappa_s^2 + H_2 \kappa_d^2 + H_3 \kappa_{sh}^2 \right).$$

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- It is important to mention that in this theory both torsion and nonmetricity are propagating and moreover, their spin-2 and spin-3 parts are dynamical.
- All the masses of the tensor modes of torsion and nonmetricity are different from zero \implies We evaded the Weinberg-Witten no-go theorem (massless higher-spin fields are pathological)

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$$\#20(R_{\lambda\rho\mu\nu}) = \#10(W_{\lambda\rho\mu\nu}) + \#9(\mathcal{R}_{\rho\nu}) + \#1(R).$$

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- **Result in Riemannian geometry:** Weyl has 6 types (**Petrov classification**); Ricci traceless has 15 types (**Segre classification**);
 - **What happens in GR in vacuum?** $\mathcal{R}_{\rho\nu} = R = 0$ and then the curvature is fully characterised by the Weyl tensor with their 6 types.

- This classification can be derived by means of its principal null directions which requires expressing any tensor in terms of a set of null vectors l_μ , k_μ , m_μ , and \bar{m}_μ ;

$$\begin{aligned}k^\mu l_\mu &= -m^\mu \bar{m}_\mu = 1, \\k^\mu m_\mu &= k^\mu \bar{m}_\mu = l^\mu m_\mu = l^\mu \bar{m}_\mu = 0, \\k^\mu k_\mu &= l^\mu l_\mu = m^\mu m_\mu = \bar{m}^\mu \bar{m}_\mu = 0.\end{aligned}$$

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Algebraic type	Segre characteristic	Intrinsic characterisation
I	[1 1 1]	$l_{[\sigma} {}^{(1)}\tilde{W}_{\lambda]\rho\mu[\nu} l_\omega] l^\rho l^\mu = 0$
II	[2 1]	${}^{(1)}\tilde{W}_{\lambda\rho\mu[\nu} l_\omega] l^\rho l^\mu = 0$
D	[(1 1) 1]	${}^{(1)}\tilde{W}_{\lambda\rho\mu[\nu} k_\omega] k^\rho k^\mu = {}^{(1)}\tilde{W}_{\lambda\rho\mu[\nu} l_\omega] l^\rho l^\mu = 0$
III	[3]	${}^{(1)}\tilde{W}_{\lambda\rho\mu[\nu} l_\omega] l^\mu = 0$
N	[(2 1)]	${}^{(1)}\tilde{W}_{\lambda\rho\mu\nu} l^\mu = 0$
O	[-]	${}^{(1)}\tilde{W}_{\lambda\rho\mu\nu} = 0$

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- The most general Type D solution in Einstein-Maxwell is known as the Plebański-Demiański characterised by $\{M, a, \alpha, N\}$ (mass, angular momentum, acceleration and Nut charge) and the electromagnetic charges.
- *Goldberg-Sachs theorem*: A vacuum solution of the Einstein's field equations admits a shear-free null geodesic congruence if and only if the conformal part of the Riemann tensor is algebraically special.

Curvature decomposition in Metric-Affine geometry

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- We find that there are 11 building blocks S. Bahamonde and J. Gigante Valcarcel, Phys. Rev. D **108** (2023) no.4, 4:

Building block	Number of independent components	Limit in Riemannian geometry
${}^{(1)}\tilde{Z}_{\lambda\rho\mu\nu}$	30	zero
${}^{(1)}\tilde{W}_{\lambda\rho\mu\nu}$	10	Weyl tensor $W_{\lambda\rho\mu\nu}$
$\tilde{R}_{\lambda[\rho\mu\nu]}^{(T)}$	9	zero
$\tilde{R}_{\lambda[\rho\mu\nu]}^{(Q)}$	9	zero
$\tilde{R}_{(\mu\nu)}$	9	Ricci traceless $\hat{R}_{\mu\nu}$
$\tilde{R}_{(\mu\nu)}^{(Q)}$	9	zero
$\tilde{R}_{[\mu\nu]}^{(T)}$	6	zero
$\tilde{R}_{[\mu\nu]}^{(Q)}$	6	zero
$\tilde{R}^{\lambda}_{\lambda\mu\nu}$	6	zero
\tilde{R}	1	Ricci scalar R
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 - This tensor is very difficult to classify since it contains 30 degrees of freedom (dof) with the following symmetry properties:

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- 3 It is common that in spherical symmetry, the field strength tensors are of Type D (two null directions aligned). For a black hole solution endowed with shears, we found that even in spherical symmetry, ${}^{(1)}\tilde{Z}^{\lambda}{}_{\lambda\mu\nu}$ is no longer Type D.

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- The general MAG is a gauge theory of gravity where torsion, nonmetricity and curvature appear as field strength tensors.
- Within MAG, we constructed the cosmological perturbations where torsion/nonmetricity can be propagating.

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- Due to the difficulty of these theories, it is important to understand certain symmetries or special important cases for the fields. To understand this, we formulated the algebraic classification of MAG for the curvature tensor. This is analogous to the Petrov classification.

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- It is possible to find a renormalizable theory with these cubic interactions?