Introduction to Metric-Affine Gravity

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How to modify GR?

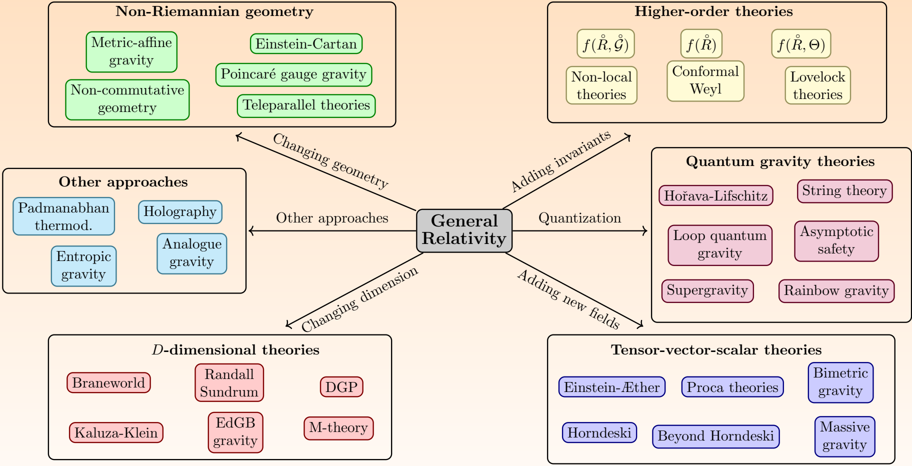

Figure: Classification of theories of gravity. (S. Bahamonde, K. F. Dialektopoulos, C. Escamilla-Rivera, G. Farrugia, V. Gakis, M. Hendry, M. Hohmann, J. Levi Said, J. Mifsud and E. Di Valentino, “Teleparallel gravity: from theory to cosmology,” Rept. Prog. Phys. **86** (2023) no.2, 026901.)

Definitions and conventions

- The affine/linear connection $\tilde{\Gamma}^{\rho}{}_{\lambda\mu}$ defines the covariant derivative operator

$$\tilde{\nabla}_{\mu}V^{\nu} = \partial_{\mu}V^{\nu} + \tilde{\Gamma}^{\nu}{}_{\lambda\mu}V^{\lambda}\,,$$
$$\tilde{\nabla}_{\mu}V^{\nu} \xrightarrow{GCT} \tilde{\nabla}'_{\mu}V'^{\nu} = \Lambda_{\mu}{}^{\lambda}\Lambda^{\nu}{}_{\rho}\tilde{\nabla}_{\lambda}V^{\rho}$$

and provides the notion of parallel transport along a curve

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- In general, the metric tensor and the affine connection are independent quantities.

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- Definition of the curvature tensor:

$$\tilde{R}^{\lambda}{}_{\rho\mu\nu} = \partial_{\mu}\tilde{\Gamma}^{\lambda}{}_{\rho\nu} - \partial_{\nu}\tilde{\Gamma}^{\lambda}{}_{\rho\mu} + \tilde{\Gamma}^{\lambda}{}_{\sigma\mu}\tilde{\Gamma}^{\sigma}{}_{\rho\nu} - \tilde{\Gamma}^{\lambda}{}_{\sigma\nu}\tilde{\Gamma}^{\sigma}{}_{\rho\mu} \,.$$

- In particular, it measures the change of vector components on parallel transport along an infinitesimal closed curve:

$$\delta V_{\mu} = \tilde{R}^{\lambda}{}_{\mu\rho\nu} V_{\lambda} ds^{\rho\nu} \,,$$

where $ds^{\rho\nu}$ denotes the surface element spanned by the infinitesimal closed curve.

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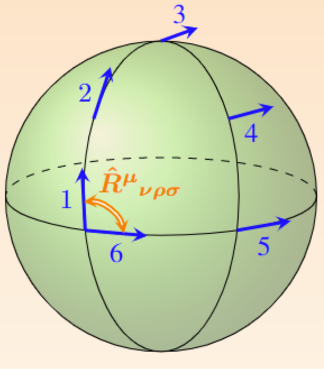

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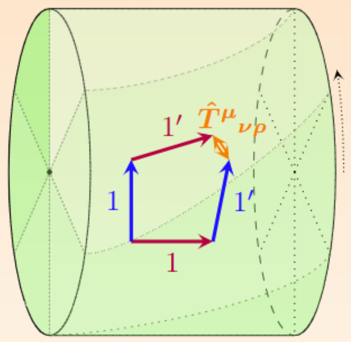
- Although the affine connection is not a tensor quantity, its antisymmetric part transforms as a tensor under general coordinate transformations.
- In particular, it measures the nonclosure of infinitesimal parallelograms:

$$\delta_{U,V} = T^{\rho}{}_{\mu\nu} \delta U^{\mu} \delta V^{\nu} \, .$$

- 24 independent components.



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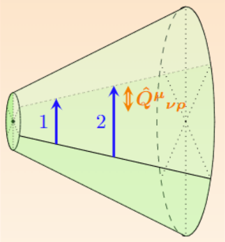

Post-Riemannian decomposition

- In the most general metric-affine setting, the fundamental variables are a **metric** $g_{\mu\nu}$ (10 comp.) as well as the coefficients $\tilde{\Gamma}^{\rho}{}_{\mu\nu}$ (64 comp.) of an **affine connection**.

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- Further, we can write

$$\tilde{\Gamma}^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\mu\nu} + K^{\lambda}{}_{\mu\nu} + L^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\mu\nu} + N^{\lambda}{}_{\mu\nu}$$
$$\tilde{R}^{\lambda}{}_{\rho\mu\nu} = R^{\lambda}{}_{\rho\mu\nu} + 2\nabla_{[\mu|}N^{\lambda}{}_{\rho|\nu]} + 2N^{\lambda}{}_{\sigma[\mu|}N^{\sigma}{}_{\rho|\nu]} \,.$$

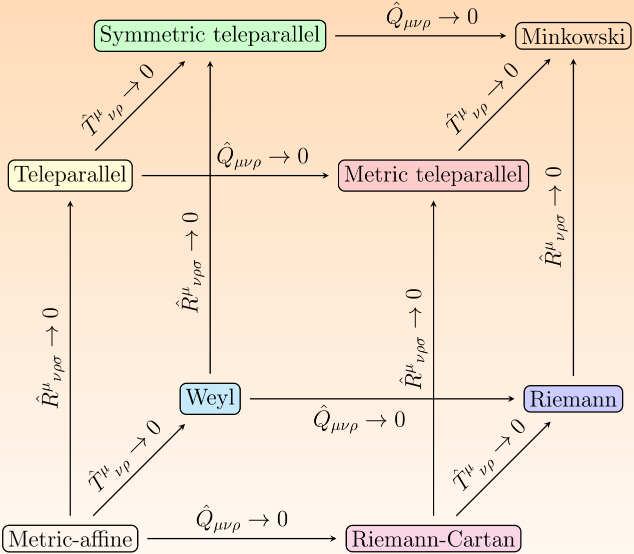

Figure: Classification of metric-affine geometries - Cube (S. Bahamonde, K. F. Dialektopoulos, C. Escamilla-Rivera, G. Farrugia, V. Gakis, M. Hendry, M. Hohmann, J. Levi Said, J. Mifsud and E. Di Valentino, "Teleparallel gravity: from theory to cosmology," Rept. Prog. Phys. **86** (2023) no.2, 026901.)

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$$\begin{aligned}
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\tilde{\nabla}_{[\sigma|}\tilde{R}^{\lambda}{}_{\rho|\mu\nu]} &= T^{\omega}{}_{[\sigma\mu|}\tilde{R}^{\lambda}{}_{\rho\omega|\nu]}\,, \\
\tilde{R}^{(\lambda\rho)}{}_{\mu\nu} &= \tilde{\nabla}_{[\nu}Q_{\mu]}{}^{\lambda\rho} + \frac{1}{2}T^{\sigma}{}_{\mu\nu}Q_{\sigma}{}^{\lambda\rho}\,.
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$$\tilde{R}^{(\lambda\rho)}{}_{\mu\nu} = \tilde{\nabla}_{[\nu} Q_{\mu]}{}^{\lambda\rho} + \frac{1}{2} T^{\sigma}{}_{\mu\nu} Q_{\sigma}{}^{\lambda\rho}\,.$$

- Three independent second order tensors defined from the first contractions of the curvature tensor:

$$\tilde{R}_{\mu\nu} = \tilde{R}^{\lambda}{}_{\mu\lambda\nu}\,, \quad \hat{R}_{\mu\nu} = \tilde{R}_{\mu}{}^{\lambda}{}_{\nu\lambda}\,, \quad \tilde{R}^{\lambda}{}_{\lambda\mu\nu} = \nabla_{[\nu} Q_{\mu]\lambda}{}^{\lambda}\,.$$

- Due to torsion, this connection introduces modifications in the covariant derivative which indeed involves a change on its commutation relations when considering an arbitrary vector v^λ:

$$[\tilde{\nabla}_\mu, \tilde{\nabla}_\nu]\, v^\lambda = \tilde{R}^\lambda{}_{\rho\mu\nu}\, v^\rho + T^\rho{}_{\mu\nu}\, \tilde{\nabla}_\rho v^\lambda\,.$$

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- The change of lengths of a given vector k^μ as well as the change of angles between two unit timelike vectors $\hat{m}^\mu$ and $\hat{n}^\mu$, under a general parallel transport defined by a tangent vector V^μ, is proportional to the nonmetricity tensor:

$$V^\lambda \tilde{\nabla}_\lambda ||\mathbf{k}||^2 = V^\lambda Q_{\lambda\mu\nu} k^\mu k^\nu\,,$$

$$V^\lambda \tilde{\nabla}_\lambda \left(g_{\mu\nu} \hat{m}^\mu \hat{n}^\nu\right) = V^\lambda Q_{\lambda\mu\nu} \hat{m}^\mu \hat{n}^\nu - \frac{1}{2} V^\lambda Q_{\lambda\mu\nu} \left(\hat{m}^\mu \hat{m}^\nu + \hat{n}^\mu \hat{n}^\nu\right) \hat{m}^\rho \hat{n}_\rho\,.$$

Dirac equation in the presence of torsion

- Covariant derivative of a Dirac spinor and its adjoint

$$\tilde{\nabla}_\mu \Psi = \partial_\mu \Psi - \frac{1}{8}\,\omega^{ab}{}_{\mu}\,[\gamma_a, \gamma_b]\,\Psi\,,$$
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- The Dirac Lagrangian provides the dynamics of $1/2$ spin fields under minimal coupling

$$\tilde{\mathcal{L}}_{\rm Dirac} = \frac{i}{2}\left(\bar{\Psi}\,\gamma^\mu \tilde{\nabla}_\mu \Psi - \tilde{\nabla}_\mu \bar{\Psi}\,\gamma^\mu \Psi - 2im\bar{\Psi}\Psi\right)\,.$$

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$$\tilde{\mathcal{L}}_{\rm Dirac} = \mathcal{L}_{Dirac} - \frac{i}{16} e^{a\mu} e^{b}{}_{\lambda} e^{c\rho} K^{\lambda}{}_{\rho\mu} \bar{\Psi}\,\{\gamma_a, [\gamma_b, \gamma_c]\}\,\Psi\,.$$

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- Accordingly, only the axial mode of torsion interacts with $1/2$ fields under minimal coupling

$$\tilde{\mathcal{L}}_{Dirac} = \mathcal{L}_{Dirac} - \frac{1}{8}\,\varepsilon^{\lambda\rho\mu\nu} T_{\lambda\rho\mu} \bar{\Psi}\gamma^5 \gamma_\nu \Psi\,.$$

- Let us consider the Generalized Einstein-Hilbert action without nonmetricity:

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What happens to the Einstein-Hilbert action if we upgrade it to $\tilde{R}$? - vacuum

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- By varying the above action with respect to Γ we find

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- By solving this equation we find that $\tilde{\Gamma} = \Gamma$! (the Christoffel symbols - Levi-Civita connection)
- This means that from the Einstein-Hilbert action in vacuum, we do not need to assume the connection - it is chosen to be the Levi-Civita one.

Simplest modify theory - Einstein-Cartan gravity

- The simplest modification of GR is the same theory as before but with matter:

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- Now by taking variations the connection eq is modified:

$$T^{\mu}{}_{\alpha\beta} + 2 g^{\mu}_{[\alpha} T^{\lambda}{}_{\beta]\lambda} = \kappa \Delta^{\mu}{}_{\alpha\beta} ,$$

with

$$\Delta^{\mu}{}_{\alpha}{}^{\beta} := \frac{1}{\sqrt{-g}} \frac{\delta(\sqrt{-g} L_{\mathrm{m}})}{\delta \Gamma^{\alpha}{}_{\beta\mu}}$$

is the so-called hypermomentum - in this case tensor of spin.

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- By varying with respect to the metric and using the connection eq:
$$G_{\mu\nu} = \kappa T_{\mu\nu} + \kappa^2\Big\{ -4\Delta_{\mu\lambda}{}^{\alpha}\Delta_{\nu\alpha}{}^{\lambda]} - 2\Delta_{\mu\lambda\alpha}\Delta_{\nu}{}^{\lambda\alpha}$$
$$+\Delta_{\alpha\lambda\mu}\Delta^{\alpha\lambda}{}_{\nu} + \frac{1}{2}g_{\mu\nu}\big(4\Delta_{\lambda}{}^{\beta}{}_{[\alpha}\Delta^{\lambda\alpha}{}_{\beta]} + \Delta^{\alpha\lambda\beta}\Delta_{\alpha\lambda\beta}\big)\Big\}\,.$$

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- How to select the action? **Gauge approach to gravity is a way**

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- Then, a gauge connection of the Poincaré group $ISO(1,3)$ can be introduced to describe the gravitational field as a gauge field of the external rotations and translations.
- A nontrivial spin density tensor which operates as source of torsion$\Longrightarrow$ intrinsic spin generates gravity!

Gauge formalism of metric-affine geometry

- The general case does not assume anything so one has a manifold with curvature, torsion and nonmetricity.

[1] F. W. Hehl, J. D. McCrea, E. W. Mielke and Y. Ne'eman, Phys. Rept. **258**, 1 (1995).

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- Affine group $A(4,\mathcal{R}) = R^4 \otimes GL(4,R)$ is the semiproduct of the translation group R^4 and the general linear group $GL(4,R)$. gauge connection with an independent local metric structure[1]:

$$A_\mu = e^a{}_\mu P_a + \omega^a{}_{b\mu} L_a{}^b \,,$$
$$g_{\mu\nu} = e^a{}_\mu e^b{}_\nu \eta_{ab} \,.$$

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- Generators of the group $A(4,\mathcal{R})$:

$$[P_a, P_b] = 0 \,,$$
$$\left[L_a{}^b, P_c\right] = i\,\delta^b{}_c\,P_a \,,$$
$$\left[L_a{}^b, L_c{}^d\right] = i\left(\delta^b{}_c\,L_a{}^d - \delta_a{}^d\,L_c{}^b\right) \,.$$

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\begin{aligned}
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F^a{}_{\mu\nu} &= \partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu + \omega^a{}_{b\mu}\,e^b{}_\nu - \omega^a{}_{b\nu}\,e^b{}_\mu\,,\\
F^a{}_{b\mu\nu} &= \partial_\mu \omega^a{}_{b\nu} - \partial_\nu \omega^a{}_{b\mu} + \omega^a{}_{c\mu}\,\omega^c{}_{b\,\nu} - \omega^a{}_{c\nu}\,\omega^c{}_{b\mu}\,.
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- it is possible to obtain the following gauge curvatures from the anholonomic metric, coframe and connection:

$$G_{ab\mu} = \partial_\mu g_{ab} - g_{ac}\,\omega^c{}_{b\mu} - g_{bc}\,\omega^c{}_{a\mu}\,,$$
$$F^a{}_{\mu\nu} = \partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu + \omega^a{}_{b\mu}\,e^b{}_\nu - \omega^a{}_{b\nu}\,e^b{}_\mu\,,$$
$$F^a{}_{b\mu\nu} = \partial_\mu \omega^a{}_{b\nu} - \partial_\nu \omega^a{}_{b\mu} + \omega^a{}_{c\mu}\,\omega^c{}_{b\,\nu} - \omega^a{}_{c\nu}\,\omega^c{}_{b\mu}\,.$$

- Correspondence with the curvature, torsion and nonmetricity tensors:

$$\begin{aligned} G_{ab\mu} &= g_{ac}g_{bd}e^{c\lambda}e^{d\rho}Q_{\mu\lambda\rho}, \\ F^a{}_{\mu\nu} &= e^a{}_\lambda T^\lambda{}_{\nu\mu}\,, \\ F^a{}_{b\mu\nu} &= g_{bc}\,e^a{}_\lambda e^{c\rho}\tilde{R}^\lambda{}_{\rho\mu\nu}\,. \end{aligned}$$

Dynamics of metric-affine geometry

- Gravitational action with dynamical torsion and nonmetricity:

$$S = \int d^4x \sqrt{-g} \left[\mathcal{L}_m - \frac{1}{16\pi} \mathcal{L}_g(\tilde{\mathcal{R}}, \mathcal{T}, \mathcal{Q}) \right] .$$

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- Correspondence between geometry and matter:

$$\begin{aligned} \frac{\delta S_g}{\delta e^a{}_\nu} &= 16\pi\theta_a{}^\nu , \\ \frac{\delta S_g}{\delta\omega^a{}_{b\nu}} &= 16\pi\Delta_a{}^{b\nu} . \end{aligned}$$

Here $\theta_a{}^\nu$ is the energy-momentum tensor (canonical) and $\Delta_a{}^{b\nu}$ is the hypermomentum density tensor.

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- $GL(4, R)$ group allows the definition of a large number of scalar invariants depending on the aforementioned tensors.

Dynamics of Quadratic Poincare gauge gravity

- By considering all possible scalars that are invariant under the Poincare group, one gets the following quadratic action:

$$S=\frac{1}{2\kappa}\int d^4x\sqrt{-g}\Big[a_0\tilde{R}+2\Lambda_0+\beta_1\,\tilde{R}_{\alpha\beta\mu\nu}\tilde{R}^{\alpha\beta\mu\nu}+\beta_2\,\tilde{R}_{\alpha\beta\mu\nu}\tilde{R}^{\alpha\mu\beta\nu}$$
$$+\beta_3\,\tilde{R}_{\alpha\beta\mu\nu}\tilde{R}^{\mu\nu\alpha\beta}+\beta_4\,\tilde{R}_{\alpha\beta}\tilde{R}^{\alpha\beta}+\beta_5\,\tilde{R}_{\alpha\beta}\tilde{R}^{\beta\alpha}+\beta_6\,\tilde{R}^2$$
$$+\alpha_1\,T_{\alpha\beta}{}^{\nu}\,T^{\alpha\beta}{}_{\nu}+\alpha_2\,T_{\alpha}\,T^{\alpha}+\alpha_3\,T_{\alpha\beta}{}^{\nu}\,T_{\nu}{}^{\alpha\beta}\Big]$$

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$$+\,\beta_3\,\tilde{R}_{\alpha\beta\mu\nu}\tilde{R}^{\mu\nu\alpha\beta} + \,\beta_4\,\tilde{R}_{\alpha\beta}\tilde{R}^{\alpha\beta} + \beta_5\,\tilde{R}_{\alpha\beta}\tilde{R}^{\beta\alpha} + \beta_6\,\tilde{R}^2$$

$$+\,\alpha_1\,T_{\alpha\beta}{}^{\nu}\,T^{\alpha\beta}{}_{\nu} + \alpha_2\,T_\alpha\,T^\alpha + \alpha_3\,T_{\alpha\beta}{}^{\nu}\,T_\nu{}^{\alpha\beta}\Big]$$

- This theory is called ”Quadratic Poincare Gauge Gravity theory” and in general, propagates ghosts.