

# Cosmological Perturbation Theory in Metric-Affine Gravity

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## Abstract

We formulate cosmological perturbation theory around the spatially curved FLRW background in the framework of Metric-Affine Gravity (MAG). Performing scalar-vector-tensor decomposition of the torsion and nonmetricity perturbations, we find that the theory displays a rich perturbation spectrum with helicities 0, 1, 2 and 3, on top of the usual scalar, vector and tensor metric perturbations arising from Riemannian geometry, which in general provides a diverse phenomenology. As an immediate application, we analyse the linear perturbations of the nonmetricity helicity-3 modes for the general parity preserving action of quadratic MAG, obtaining the conditions to avoid possible instabilities in this sector of the theory.

## 1. What is Metric-Affine Gravity?

- It is a theory of gravity based on a Lorentzian manifold with **independent metric tensor**  $g_{\mu\nu}$  and **affine connection**  $\tilde{\Gamma}^\lambda_{\mu\nu}$ .

- The affine connection includes torsion and nonmetricity **as its antisymmetric part and as the covariant derivative of the metric tensor**

$$T^\lambda_{\mu\nu} = 2\tilde{\Gamma}^\lambda_{[\mu\nu]}, \quad Q_{\lambda\mu\nu} = \tilde{\nabla}_\lambda g_{\mu\nu}, \quad (1)$$

respectively.

- Although the Levi-Civita affine connection does not transform covariantly under diffeomorphisms, **torsion and nonmetricity constitute 3-rank tensors**.

- As tensor quantities, they can be further **decomposed into irreducible parts under the pseudo-orthogonal group**, which in the four-dimensional case is displayed as follows

$$T^\lambda_{\mu\nu} = \frac{1}{3}(\delta^\lambda_\nu T_\mu - \delta^\lambda_\mu T_\nu) + \frac{1}{6}\varepsilon^\lambda_{\rho\mu\nu}S^\rho + t^\lambda_{\mu\nu}, \quad (2)$$

$$Q_{\lambda\mu\nu} = g_{\mu\nu}W_\lambda + g_{\lambda(\mu}A_{\nu)} - \frac{1}{4}g_{\mu\nu}A_\lambda + \frac{1}{3}\varepsilon_{\lambda\rho\sigma(\mu}Q_{\nu)}^{\rho\sigma} + q_{\lambda\mu\nu}, \quad (3)$$

where

$$T_\mu = T^\nu_{\mu\nu} \text{ is a vector,} \quad (4)$$

$$S_\mu = \varepsilon_{\mu\lambda\rho\nu}T^{\lambda\rho\nu} \text{ is a pseudovector,} \quad (5)$$

$$t_{\lambda\mu\nu} = T_{\lambda\mu\nu} - \frac{2}{3}g_{\lambda[\nu}T_{\mu]} - \frac{1}{6}\varepsilon_{\lambda\rho\mu\nu}S^\rho \text{ is a traceless and pseudotraceless tensor,} \quad (6)$$

$$W_\mu = \frac{1}{4}Q_{\mu\nu}{}^\nu \text{ is a vector,} \quad (7)$$

$$A_\mu = \frac{4}{9}(Q^\nu_{\mu\nu} - W_\mu) \text{ is a vector,} \quad (8)$$

$$\Omega_\lambda{}^{\mu\nu} = \varepsilon^{\mu\nu\rho\sigma}Q_{\sigma\rho\lambda} + \varepsilon_\lambda{}^{\mu\nu\rho}\left(\frac{3}{4}A_\rho - W_\rho\right) \text{ is a traceless and pseudotraceless tensor,} \quad (9)$$

$$q_{\lambda\mu\nu} = Q_{(\lambda\mu\nu)} - g_{(\mu\nu}W_{\lambda)} - \frac{3}{4}g_{(\mu\nu}A_{\lambda)} \text{ is a fully symmetric tensor.} \quad (10)$$

- In the affine connection, the torsion and nonmetricity tensors are displayed as

$$\tilde{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + \frac{1}{2}(T^\lambda_{\mu\nu} - T_\mu{}^\lambda{}_\nu - T_\nu{}^\lambda{}_\mu + Q^\lambda_{\mu\nu} - Q_\mu{}^\lambda{}_\nu - Q_\nu{}^\lambda{}_\mu), \quad (11)$$

which provides a **curvature tensor** of the form

$$\tilde{R}^\lambda_{\rho\mu\nu} = \partial_\mu\tilde{\Gamma}^\lambda_{\rho\nu} - \partial_\nu\tilde{\Gamma}^\lambda_{\rho\mu} + \tilde{\Gamma}^\lambda_{\sigma\mu}\tilde{\Gamma}^\sigma_{\rho\nu} - \tilde{\Gamma}^\lambda_{\sigma\nu}\tilde{\Gamma}^\sigma_{\rho\mu}. \quad (12)$$

- From a physical point of view, the **gravity sector** based on metric-affine geometry is described by an **action functional defined from the curvature, torsion and nonmetricity invariants**

$$S_g = \int \mathcal{L}_g(\tilde{R}, \mathcal{J}, \Omega)\sqrt{-g}d^4x, \quad (13)$$

whereas the **matter sector** includes general couplings with the metric tensor and the affine connection, which provides **canonical energy-momentum and hypermomentum tensors**

$$\theta_{\mu\nu} = \frac{1}{\sqrt{-g}}\frac{\delta(\mathcal{L}_m\sqrt{-g})}{\delta g^{\mu\nu}}, \quad \Delta_\lambda{}^{\mu\nu} = \frac{1}{\sqrt{-g}}\frac{\delta(\mathcal{L}_m\sqrt{-g})}{\delta\tilde{\Gamma}^\lambda_{\mu\nu}}. \quad (14)$$

- The theory gives rise to a large number of phenomenological implications at astrophysical and cosmological scales, while a **thorough study on its cosmological perturbations is required to perform any realistic model for the description of the inhomogeneities and anisotropies of the universe, such as the structure formation at large scales, the primordial fluctuations in the CMB and the propagation of gravitational waves at cosmological distances**, among others.

## 2. 3+1 and helicity decompositions of the metric, torsion and nonmetricity tensors

- In order to find the spectrum of cosmological perturbations in MAG, **we first set up the cosmological background**, which consists of homogeneous and isotropic metric, torsion and nonmetricity tensors.

- The metric tensor is described then by the line element of the **FLRW space-time**

$$ds^2 = -N^2(t)dt^2 + a^2(t)\gamma_{ij}dx^i dx^j = -N^2(t)dt^2 + a^2(t)\left(\frac{dr^2}{1-Kr^2} + r^2d\Omega^2\right), \quad (15)$$

while the torsion and nonmetricity tensors acquire the form

$$\tilde{T}^\lambda_{\mu\nu} = 2T_1(t)\tilde{n}_{[\mu}\tilde{P}_{\nu]}^\lambda + 2T_2(t)\tilde{\varepsilon}^\lambda{}_{\mu\nu\rho}\tilde{n}^\rho, \quad (16)$$

$$\tilde{Q}_{\lambda\mu\nu} = 2Q_1(t)\tilde{n}_\lambda\tilde{n}_\mu\tilde{n}_\nu + 2Q_2(t)\tilde{n}_\lambda\tilde{P}_{\mu\nu} + 2Q_3(t)\tilde{P}_{\lambda(\mu}\tilde{n}_{\nu)}, \quad (17)$$

where  $\tilde{n}_\mu$  is a unit timelike vector and  $\tilde{P}_{\mu\nu}$  a projection tensor, satisfying  $\tilde{P}_{\mu\nu}\tilde{n}^\nu = 0$ .

- On top of this cosmological background, the perturbed metric, torsion and nonmetricity tensors are then decomposed into irreducible parts under the three-dimensional special orthogonal group, which constitutes the **3+1 decomposition** of these tensors.

- For the metric tensor, this algebraic procedure gives rise to two scalar quantities, one spatial vector and one spatial traceless tensor

$$\delta g_{\mu\nu} = -2\tilde{n}_\mu\tilde{n}_\nu\alpha + 2\tilde{P}_{\mu\nu}\psi - 2\tilde{n}_{(\mu}\tilde{\beta}_{\nu)} + \tilde{h}_{\mu\nu}. \quad (18)$$

- The perturbed vector, pseudovector and tensor parts of the torsion tensor are reducible to further scalars, spatial vectors and pseudovectors, as well as spatial tensors and pseudotensors

$$\delta T_\mu = \tilde{T}_\mu - \tilde{n}_\mu\phi, \quad \delta S_\mu = \tilde{S}_\mu - \tilde{n}_\mu\rho, \quad (19)$$

$$\delta t_{\lambda\mu\nu} = 2\tilde{n}_{[\nu}\tilde{A}_{\rho]\mu} + 2(\tilde{n}_\mu\tilde{n}_{[\nu} + \frac{1}{2}\tilde{P}_{\mu[\nu}]\tilde{\beta}_{\rho]})\tilde{B}_{\rho]} + \frac{1}{2}\varepsilon_{\nu\rho}{}^{\alpha\beta}[-\tilde{n}_\alpha\tilde{A}_{\beta\mu} + (\tilde{n}_\beta\tilde{n}_\mu + \frac{1}{2}\tilde{P}_{\beta\mu})\tilde{B}_\alpha]. \quad (20)$$

- The richer algebraic structure of the nonmetricity tensor provides a higher number of d.o.f.

$$\delta W_\mu = \tilde{W}_\mu - \tilde{n}_\mu\theta, \quad \delta A_\mu = \tilde{A}_\mu - \tilde{n}_\mu\sigma, \quad (21)$$

$$\delta\Omega_{\lambda\mu\nu} = 2\tilde{n}_{[\nu}\tilde{Q}_{\rho]\mu} + 2(\tilde{n}_\mu\tilde{n}_{[\nu} + \frac{1}{2}\tilde{P}_{\mu[\nu}]\tilde{\beta}_{\rho]})\tilde{Y}_{\rho]} + \frac{1}{2}\varepsilon_{\nu\rho}{}^{\alpha\beta}[-\tilde{n}_\alpha\tilde{Q}_{\beta\mu} + (\tilde{n}_\beta\tilde{n}_\mu + \frac{1}{2}\tilde{P}_{\beta\mu})\tilde{Y}_\alpha], \quad (22)$$

$$\delta q_{\lambda\mu\nu} = \tilde{C}_{\lambda\mu\nu} - 3\tilde{n}_{[\mu}\tilde{\kappa}_{\nu\rho]} + \frac{3}{5}(5\tilde{n}_{[\mu}\tilde{n}_{\nu} + \tilde{P}_{(\mu\nu)}]\tilde{Z}_{\rho]} - (\tilde{n}_\mu\tilde{n}_\nu + \tilde{P}_{(\mu\nu)})\tilde{n}_\rho\xi. \quad (23)$$

- In order to separate the different helicity modes included in the spatial parts, a **SVT or helicity decomposition is needed**

$$\delta\tilde{\mathbf{x}}_i = D_i\mathbf{S} + \mathbf{V}_i^{(T)}, \quad \delta\tilde{\mathbf{x}}_{ij} = (D_{(i}D_{j)} - \frac{1}{3}\gamma_{ij}D^2)\mathbf{S} + D_{(i}\mathbf{V}_{j)}^{(T)} + \mathbf{T}_{ij}^{(TT)}, \quad (24)$$

$$\delta\tilde{\mathbf{x}}_{ijk} = [D_{(i}D_{j}D_{k)} - \frac{1}{5}\gamma_{(ij}D_{k)}(3D^2 + 4K)]\mathbf{S} + [D_{(i}D_{j)} - \frac{1}{5}\gamma_{(ij}(D^2 + 2K)]\mathbf{V}_{k)}^{(T)} + D_{(i}\mathbf{T}_{jk)}^{(TT)} + \mathbf{TT}_{ijk}^{(TT)}, \quad (25)$$

where the labels (T) and (TT) denote the corresponding helicity modes are transverse and transverse-traceless, respectively.

- All of the helicity modes for the torsion and nonmetricity tensors are displayed in the tables:

SVT	Quantities	d.o.f.	Total d.o.f.
4 scalars	{T, B, ϕ, A}	1 d.o.f. each	4
4 pseudoscalars	{S, B, ρ, A}	1 d.o.f. each	4
3 vectors	{T <sub>i</sub> <sup>(T)</sup> , B <sub>i</sub> <sup>(T)</sup> , A <sub>i</sub> <sup>(T)</sup> }	2 d.o.f. each	6
3 pseudovectors	{S <sub>i</sub> <sup>(T)</sup> , B <sub>i</sub> <sup>(T)</sup> , A <sub>i</sub> <sup>(T)</sup> }	2 d.o.f. each	6
1 rank-2 tensor	{A <sub>ij</sub> <sup>(TT)</sup> }	2 d.o.f. each	2
1 rank-2 pseudotensor	{A <sub>ij</sub> <sup>(TT)</sup> }	2 d.o.f. each	2

Table: Helicity decomposition of the perturbations for the torsion tensor.

SVT	Quantities	d.o.f.	Total d.o.f.
10 scalars	{θ, σ, ξ, Λ, Y, Z, κ, Q, W, C}	1 d.o.f. each	10
2 pseudoscalars	{Y, Q}	1 d.o.f. each	2
7 vectors	{Λ <sub>i</sub> <sup>(T)</sup> , Y <sub>i</sub> <sup>(T)</sup> , Z <sub>i</sub> <sup>(T)</sup> , κ <sub>i</sub> <sup>(T)</sup> , Q <sub>i</sub> <sup>(T)</sup> , W <sub>i</sub> <sup>(T)</sup> , C <sub>i</sub> <sup>(T)</sup> }	2 d.o.f. each	14
2 pseudovectors	{y <sub>i</sub> <sup>(T)</sup> , q <sub>i</sub> <sup>(T)</sup> }	2 d.o.f. each	4
3 rank-2 tensor	{κ <sub>ij</sub> <sup>(TT)</sup> , Q <sub>ij</sub> <sup>(TT)</sup> , C <sub>ij</sub> <sup>(TT)</sup> }	2 d.o.f. each	6
1 rank-2 pseudotensor	{Q <sub>ij</sub> <sup>(TT)</sup> }	2 d.o.f. each	2
1 rank-3 tensor	{C <sub>ijk</sub> <sup>(TT)</sup> }	2 d.o.f. each	2

Table: Helicity decomposition of the perturbations for the nonmetricity tensor.

## 3. Perturbation analysis of the nonmetricity helicity-3 modes in quadratic MAG

- While the metric, torsion and nonmetricity tensors include a large number of d.o.f., the symmetries of the FLRW space-time allow us to significantly simplify the problem: **spatial objects with different helicities evolve in a completely independent way at the linear level of perturbations**.

- In particular, **the helicity-3 perturbation of the nonmetricity tensor decouples from all of the modes with different helicities at the linear order of perturbations**, which especially simplifies its perturbation analysis.

- For this analysis, **we consider the framework of quadratic MAG**, characterised by a gravitational action which depends on curvature, torsion and nonmetricity invariants up to quadratic order:

$$S_g = \frac{1}{16\pi} \int \left( \tilde{R} + a_1\tilde{R}^2 + a_2\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\lambda\rho\mu\nu} + a_3\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\rho\lambda\mu\nu} + a_4\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\mu\lambda\rho\nu} + a_5\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\nu\lambda\rho\mu} \right. \\ \left. + a_6\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\mu\nu\lambda\rho} + a_7\tilde{R}_{\rho\lambda\mu\nu}\tilde{R}^{\mu\lambda\rho\nu} + a_8\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\rho\lambda\mu\nu} + a_9\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\nu\lambda\rho\mu} + a_{10}\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\mu\nu\lambda\rho} + a_{11}\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\rho\mu\lambda\nu} \right. \\ \left. + a_{12}\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\nu\mu\lambda\rho} + a_{13}\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\rho\nu\lambda\mu} + a_{14}\tilde{R}^\lambda{}_{\lambda\rho\mu\nu}\tilde{R}^{\rho\mu\nu\lambda} + a_{15}\tilde{R}^\lambda{}_{\lambda\rho\mu\nu}\tilde{R}^{\mu\nu\rho\lambda} + a_{16}\tilde{R}^\lambda{}_{\lambda\rho\mu\nu}\tilde{R}^{\nu\lambda\rho\mu} \right. \\ \left. + b_1T_{\lambda\mu\nu}T^{\lambda\mu\nu} + b_2T_{\lambda\mu\nu}T^{\mu\lambda\nu} + b_3T^\lambda{}_{\lambda\nu}T^{\mu\nu} + c_1T_{\lambda\mu\nu}Q^{\lambda\mu\nu} + c_2T^\lambda{}_{\lambda\nu}Q^{\mu\nu} + c_3T^\lambda{}_{\lambda\nu}Q^{\mu\nu}{}_\mu \right. \\ \left. + d_1Q_{\lambda\mu\nu}Q^{\lambda\mu\nu} + d_2Q_{\lambda\mu\nu}Q^{\mu\lambda\nu} + d_3Q^\lambda{}_{\lambda\nu}Q^{\mu\nu} + d_4Q^\lambda{}_{\lambda\nu}Q^{\mu\nu}{}_\mu + d_5Q^\lambda{}_{\lambda\nu}Q^{\mu\nu}{}_\mu \right)\sqrt{-g}d^4x. \quad (26)$$

- For the sake of simplicity, we will concentrate on the spatially flat  $K = 0$  FLRW background and fix the cosmic time  $N(t) = 1$ .

- The **helicity-3 sector of the quadratic action** takes then the form

$$S_C = \frac{1}{16\pi} \int \mathcal{G}_T^{(C)} a^2(t) \left[ (C_{ijk}^{(TT)})^2 - \frac{1}{a^2(t)} (D_i C_{ijk}^{(TT)})^2 + \frac{4T_2(t)}{a(t)} \varepsilon_{ijk} C^{(TT)j}{}_{lm} D^l C^{(TT)km} - m_{C,\text{eff}}^2 (C_{ijk}^{(TT)})^2 \right] d^3x dt. \quad (27)$$

- In **Fourier space**:

$$C_{ijk}^{(TT)}(t, \vec{x}) = \frac{1}{(2\pi)^3} \int \frac{1}{\sqrt{2}} [C^{(L)}(t, \vec{k}) E_{ijk}^{(L)}(\vec{x}, \vec{k}) + C^{(R)}(t, \vec{k}) E_{ijk}^{(R)}(\vec{x}, \vec{k})] d^3k + \text{c.c.}, \quad (28)$$

where

$$D^2 E_{ijk}^{(L)} = -k^2 E_{ijk}^{(L)}, \quad \varepsilon_{i}{}^{pq} D_p E_{qjk}^{(L)} = -k E_{ijk}^{(L)}, \quad (29)$$

$$D^2 E_{ijk}^{(R)} = -k^2 E_{ijk}^{(R)}, \quad \varepsilon_{i}{}^{pq} D_p E_{qjk}^{(R)} = k E_{ijk}^{(R)}, \quad (30)$$

the helicity-3 sector of the quadratic action simply reads:

$$S_C = \frac{1}{16\pi} \sum_{A=L,R} \int \mathcal{G}_T^{(C)} a^2(t) \left\{ [C^{(A)}]^2 - \left( \frac{k^2}{a^2(t)} - \frac{4\lambda_A k T_2(t)}{a(t)} + m_{C,\text{eff}}^2(t) \right) [C^{(A)}]^2 \right\} \frac{d^3k}{(2\pi)^3} dt, \quad (31)$$

with  $\lambda_L = -1$ ,  $\lambda_R = +1$ .

- From the action of the helicity-3 modes, it is straightforward to obtain the **stability conditions**:

- Ghost-free kinetic terms** require  $\mathcal{G}_T^{(C)} > 0$ .
- Avoidance of gradient and tachyonic instabilities** guaranteed for  $\frac{k^2}{a^2(t)} - \frac{4\lambda_A k T_2(t)}{a(t)} + m_{C,\text{eff}}^2(t) > 0$  (note the spontaneous parity violation driven by the axial torsion at the background level might cause particle production).
- The effective mass identically vanishes in the Minkowski space-time (with vanishing torsion and nonmetricity tensors in such a space-time) if  $1 - 4(\lambda_L + \lambda_R) = 0$ , **which involves strong coupling problems and further no-go theorems that constrain the interaction of such massless modes in the quantum regime**.

- With these results, **we can systematically analyse the stability of the helicity-3 modes of the nonmetricity tensor around the spatially flat FLRW background for any gravitational model of quadratic MAG**.

- We focus on the following model of quadratic MAG with dynamical helicity 3-modes, which gives rise to **Reissner-Nordström-like black holes with spin, dilation and shear charges**:

$$S_g = \frac{1}{64\pi} \int \left[ 4R - 6\tilde{\alpha}_1 \tilde{R}_{\lambda[\rho\mu\nu]} \tilde{R}^{\lambda[\rho\mu\nu]} - 9\tilde{\alpha}_2 \tilde{R}_{\lambda[\rho\mu\nu]} \tilde{R}^{\mu\lambda\rho\nu} + 2\tilde{\alpha}_3 (\tilde{R}_{[\mu\nu]} + \tilde{R}_{[\mu\nu]}) (\tilde{R}^{[\mu\nu]} + \tilde{R}^{[\mu\nu]}) \right. \\ \left. + 18\tilde{\alpha}_4 \tilde{R}_{\lambda[\rho\mu\nu]} \tilde{R}^{\lambda[\rho\mu\nu]} - 3\tilde{\alpha}_5 \tilde{R}_{\lambda[\rho\mu\nu]} \tilde{R}^{\mu\lambda\rho\nu} + 6\tilde{\alpha}_6 \tilde{R}_{\lambda[\rho\mu\nu]} \tilde{R}^{\nu\lambda\rho\mu} + 2(2\tilde{\alpha}_7 - \tilde{f}_1) \tilde{R}^\lambda{}_{\lambda\rho\mu\nu} \tilde{R}^{\rho\mu\nu\lambda} \right. \\ \left. + 8\tilde{f}_1 \tilde{R}_{\lambda[\rho\mu\nu]} \tilde{R}^{\lambda[\rho\mu\nu]} - 2\tilde{f}_1 (\tilde{R}_{[\mu\nu]} - \tilde{R}_{[\mu\nu]}) (\tilde{R}^{[\mu\nu]} - \tilde{R}^{[\mu\nu]}) - 3(1 - 2\tilde{\alpha}_2) T_{[\lambda\mu\nu]} T^{[\lambda\mu\nu]} \right] d^4x \sqrt{-g}. \quad (32)$$

- The parameters controlling the stability of the model are

$$\mathcal{G}_T^{(C)} = -\tilde{f}_1 > 0, \quad m_{C,\text{eff}}^2(t) = \dot{H}(t) + 2H^2(t) + 4T_2^2(t) > 0. \quad (33)$$

- The helicity-3 modes do not include mass terms in the action, **which is problematic in the Minkowski limit with homogeneous and isotropic torsion and nonmetricity tensors**.

- A simple solution is found in cubic MAG**, where the model includes massive tensor modes.