Stability in Cubic Metric-Affine Gravity

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Abstract

We analyse the stability issue of the vector and axial modes of the torsion and nonmetricity tensors around general backgrounds in the framework of cubic Metric-Affine Gravity. We show that the presence of cubic order invariants defined from the curvature, torsion and nonmetricity tensors allow the cancellation of the well-known instabilities arising in the vector and axial sectors of quadratic Metric-Affine Gravity. For the resulting theory, we also obtain Reissner-Nordström-like black hole solutions with dynamical torsion and nonmetricity, which in general include massive tensor modes for these quantities, thus avoiding further no-go theorems that potentially prevent a consistent interaction of massless higher spin fields in the quantum regime.

1. Stability in quadratic MAG

- MAG constitutes an extension of General Relativity, **which accounts for the independent d.o.f. included in the affine connection of a Lorentzian manifold**.
- On top of the curvature tensor

$$\tilde{R}^{\lambda}{}_{\rho\mu\nu} = \partial_{\mu}\tilde{\Gamma}^{\lambda}{}_{\rho\nu} - \partial_{\nu}\tilde{\Gamma}^{\lambda}{}_{\rho\mu} + \tilde{\Gamma}^{\lambda}{}_{\sigma\mu}\tilde{\Gamma}^{\sigma}{}_{\rho\nu} - \tilde{\Gamma}^{\lambda}{}_{\sigma\nu}\tilde{\Gamma}^{\sigma}{}_{\rho\mu}\,, \tag{1}$$

the geometry includes the torsion and nonmetricity tensors as the antisymmetric part of the affine connection and as the covariant derivative of the metric tensor

$$T^{\lambda}{}_{\mu\nu} = 2\tilde{\Gamma}^{\lambda}{}_{[\mu\nu]}\,, \quad Q_{\lambda\mu\nu} = \tilde{\nabla}_{\lambda}g_{\mu\nu}\,, \tag{2}$$

respectively.

- **The irreducible decomposition under the pseudo-orthogonal group splits these tensors into irreducible parts**, which in the four-dimensional case are displayed as

$$T^{\lambda}{}_{\mu\nu} = \frac{1}{3}\left(\delta^{\lambda}{}_{\nu}T_{\mu} - \delta^{\lambda}{}_{\mu}T_{\nu}\right) + \frac{1}{6}\,\varepsilon^{\lambda}{}_{\rho\mu\nu}S^{\rho} + t^{\lambda}{}_{\mu\nu}\,, \tag{3}$$

$$Q_{\lambda\mu\nu} = g_{\mu\nu}W_{\lambda} + g_{\lambda(\mu}\Lambda_{\nu)} - \frac{1}{4}g_{\mu\nu}\Lambda_{\lambda} + \frac{1}{3}\varepsilon_{\lambda\rho\sigma(\mu}\Omega_{\nu)}{}^{\rho\sigma} + q_{\lambda\mu\nu}\,. \tag{4}$$

- The different d.o.f. of the metric, torsion and nonmetricity tensors, are endowed with dynamics by defining **an action functional with curvature, torsion and nonmetricity invariants**.
- The presence of the Einstein-Hilbert term in the action, which is linear in the curvature tensor, guarantees the propagation of the spin-2 field of the metric tensor, **while the introduction of higher order curvature invariants is required for the respective modes of the torsion and nonmetricity tensors to be dynamical**.
- The simplest theory with dynamical torsion and nonmetricity tensors is described by **a parity preserving action that includes invariants up to quadratic order**, which in any case produces a vast number of kinetic and interaction terms for these tensors.
- In fact, **the general action of quadratic MAG includes ghostly instabilities**, which in the theory can only be suppressed by setting strong constraints on the dynamics of these tensors.
- While it is possible to formulate quadratic MAG models with stable scalar and pseudoscalar modes of torsion and nonmetricity, **the models with dynamical vector and axial modes are generally pathological**, which at the same time affects the dynamics of the tensor modes.
- **Such models are described by a broad gravitational action**, which includes 13 coefficients associated with the quadratic curvature invariants and 11 coefficients associated with the quadratic torsion and nonmetricity invariants:

$$\begin{aligned}S_{\rm Quad} = \frac{1}{16\pi}\int\Big[&-R - \frac{1}{2}\left(2c_1 + c_2\right)\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\mu\nu\lambda\rho} + \left(a_2 - c_1\right)\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\rho\lambda\mu\nu} + a_2\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\lambda\rho\mu\nu} + a_5\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\lambda\mu\rho\nu}\\
&+ a_6\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\rho\mu\lambda\nu} + \left(c_2 - a_5 + a_6\right)\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\mu\rho\lambda\nu} + \left(d_1 - a_{10} - a_{12}\right)\tilde{R}_{\mu\nu}\tilde{R}^{\mu\nu} + a_9\tilde{R}_{\mu\nu}\tilde{R}^{\nu\mu} + a_{10}\hat{R}_{\mu\nu}\hat{R}^{\mu\nu}\\
&+ a_{11}\hat{R}_{\mu\nu}\hat{R}^{\nu\mu} - \left(d_1 + a_9 + a_{11}\right)\tilde{R}_{\mu\nu}\hat{R}^{\nu\mu} + a_{12}\tilde{R}_{\mu\nu}\hat{R}^{\mu\nu} + a_{14}\tilde{R}^{\lambda}{}_{\lambda\mu\nu}\tilde{R}^{\rho}{}_{\rho}{}^{\mu\nu} + a_{15}\tilde{R}_{\mu\nu}\tilde{R}^{\lambda}{}_{\lambda}{}^{\mu\nu} + a_{16}\hat{R}_{\mu\nu}\tilde{R}^{\lambda}{}_{\lambda}{}^{\mu\nu}\\
&+ \frac{1}{2}m_T^2T_{\mu}T^{\mu} + \frac{1}{2}m_S^2S_{\mu}S^{\mu} + \frac{1}{2}m_t^2t_{\lambda\mu\nu}t^{\lambda\mu\nu} + \frac{1}{2}m_W^2W_{\mu}W^{\mu} + \frac{1}{2}m_{\Lambda}^2\Lambda_{\mu}\Lambda^{\mu} + \frac{1}{2}m_{\Omega}^2\Omega_{\lambda\mu\nu}\Omega^{\lambda\mu\nu}\\
&+ \frac{1}{2}m_q^2q_{\lambda\mu\nu}q^{\lambda\mu\nu} + \frac{1}{2}\alpha_{TW}T_{\mu}W^{\mu} + \frac{1}{2}\alpha_{T\Lambda}T_{\mu}\Lambda^{\mu} + \frac{1}{2}\alpha_{W\Lambda}W_{\mu}\Lambda^{\mu} + \frac{1}{2}\alpha_{t\Omega}\varepsilon_{\rho\mu\nu\sigma}\Omega^{\lambda\rho\mu}t^{\nu}{}_{\lambda}{}^{\sigma}\Big]\sqrt{-g}\,d^4x\,.\end{aligned} \tag{5}$$

- In particular, **the vector and axial sectors of the theory contain Ostrogradsky instabilities**, which arise from terms in the action of the form $(\nabla X)^2$, $\nabla X\nabla Y$, $X^2\nabla Y$, $XY\nabla X$, $XY\nabla Z$, RX^2 and RXY, where $X \neq Y$ refer to the vector and axial modes T, S, W, Λ.
- Additionally, **the kinetic matrix of the vector and axial modes unavoidably presents eigenvalues with opposite signs**, thus indicating the presence of a ghost field, **unless the kinetics of the torsion modes are suppressed from the action**.
- Overall, **the cancellation of all these instabilities strongly reduces the parameter space of the quadratic curvature invariants from 13 to 5 coefficients**.

2. Cubic MAG: an example with mixing terms between curvature and torsion

- We aim to find a MAG theory with healthy vector and axial sectors, for which we consider cubic order corrections in the gravitational action.
- Concerning the curvature and torsion tensors, the most general parity preserving Lagrangian constructed from their cubic invariants **presents two different branches**: one provided by mixing terms involving these two tensors and another one by the curvature tensor alone.
- For stability purposes, **only the first branch defined from mixing terms between tensors is important to cancel out the instabilities of quadratic MAG**.
- **The corresponding Lagrangian can be in general split into six different types**, according to the possible combinations of the irreducible parts of torsion at quadratic order:

$$\mathcal{L}^{(3)}_{\rm curv-tors} = \mathcal{L}^{(3)}_{\tilde{R}TT} + \mathcal{L}^{(3)}_{\tilde{R}SS} + \mathcal{L}^{(3)}_{\tilde{R}tt} + \mathcal{L}^{(3)}_{\tilde{R}TS} + \mathcal{L}^{(3)}_{\tilde{R}Tt} + \mathcal{L}^{(3)}_{\tilde{R}St}\,, \tag{6}$$

providing 26 independent cubic invariants, which can be distributed as:

$$\mathcal{L}^{(3)}_{\tilde{R}TT} = h_1\tilde{R}_{\mu\nu}T^{\mu}T^{\nu} + h_2\tilde{R}T_{\mu}T^{\mu}\,, \tag{7}$$

$$\mathcal{L}^{(3)}_{\tilde{R}SS} = h_3\tilde{R}_{\mu\nu}S^{\mu}S^{\nu} + h_4\tilde{R}S_{\mu}S^{\mu}\,, \tag{8}$$

$$\begin{aligned}\mathcal{L}^{(3)}_{\tilde{R}tt} &= h_5\tilde{R}_{\lambda\rho\mu\nu}t_{\sigma}{}^{\lambda\rho}t^{\sigma\mu\nu} + h_6\tilde{R}_{\lambda\rho\mu\nu}t_{\sigma}{}^{\lambda\mu}t^{\sigma\rho\nu} + h_7\tilde{R}_{\lambda\rho\mu\nu}t^{\lambda\rho}{}_{\sigma}t^{\sigma\mu\nu} + h_8\tilde{R}_{\lambda\rho\mu\nu}t^{\lambda\mu}{}_{\sigma}t^{\sigma\rho\nu}\\
&+ h_9\tilde{R}_{\lambda\rho\mu\nu}t^{\lambda\mu}{}_{\sigma}t^{\rho\nu\sigma} + h_{10}\tilde{R}_{\lambda\rho}t_{\mu\nu}{}^{\lambda}t^{\rho\mu\nu} + h_{11}\tilde{R}_{\lambda\rho}t_{\mu\nu}{}^{\lambda}t^{\mu\nu\rho} + h_{12}\tilde{R}t_{\lambda\rho\mu}t^{\lambda\rho\mu}\,,\end{aligned} \tag{9}$$

$$\mathcal{L}^{(3)}_{\tilde{R}TS} = h_{13}\varepsilon^{\lambda\rho\mu\nu}\tilde{R}_{\lambda\rho\mu\nu}T_{\sigma}S^{\sigma} + h_{14}\varepsilon_{\nu}{}^{\lambda\rho\sigma}\tilde{R}_{\lambda\rho\mu\sigma}T^{\mu}S^{\nu} + h_{15}\varepsilon^{\lambda\rho\mu\nu}\tilde{R}_{\lambda\rho}T_{\mu}S_{\nu}\,, \tag{10}$$

$$\mathcal{L}^{(3)}_{\tilde{R}Tt} = h_{16}\tilde{R}_{\lambda\rho\mu\nu}T^{\nu}t^{\lambda\rho\mu} + h_{17}\tilde{R}_{\lambda\rho\mu\nu}T^{\rho}t^{\lambda\mu\nu} + h_{18}\tilde{R}_{\lambda\rho}T_{\mu}t^{\mu\lambda\rho} + h_{19}\tilde{R}_{\lambda\rho}T_{\mu}t^{\lambda\rho\mu}\,, \tag{11}$$

$$\begin{aligned}\mathcal{L}^{(3)}_{\tilde{R}St} &= h_{20}\varepsilon_{\alpha\rho\mu\nu}\tilde{R}_{\tau}{}^{\rho\mu\nu}S^{\gamma}t^{\alpha\tau}{}_{\gamma} + h_{21}\varepsilon_{\alpha\rho\mu\nu}\tilde{R}_{\tau}{}^{\rho\mu\nu}S^{\gamma}t_{\gamma}{}^{\alpha\tau} + h_{22}\varepsilon_{\alpha\rho}{}^{\mu\nu}\tilde{R}^{\rho}{}_{\mu\tau\nu}S^{\gamma}t_{\gamma}{}^{\alpha\tau}\\
&+ h_{23}\varepsilon_{\alpha\rho}{}^{\mu\nu}\tilde{R}_{\gamma\mu\tau\nu}S^{\alpha}t^{\gamma\rho\tau} + h_{24}\varepsilon_{\alpha\rho}{}^{\mu\nu}\tilde{R}_{\gamma\mu\tau\nu}S^{\alpha}t^{\rho\tau\gamma} + h_{25}\varepsilon_{\alpha\rho\tau\mu}\tilde{R}^{\mu}{}_{\gamma}S^{\alpha}t^{\rho\tau\gamma}\\
&+ h_{26}\varepsilon_{\lambda\rho\mu\nu}\tilde{R}^{\lambda\rho}S_{\sigma}t^{\sigma\mu\nu}\,.\end{aligned} \tag{12}$$

- The same procedure can be applied in the presence of nonmetricity, **giving rise to two additional branches with 61 and 102 independent cubic invariants**, respectively.

3. Stability of the vector and axial modes of torsion

- By including all the possible mixing terms of the curvature and torsion tensors of cubic order in the gravitational action, **the dependence on the vector and axial modes of torsion acquires the following form**:

$$\begin{aligned}16\pi\mathcal{L} = &-R + \frac{1}{9}\left(4c_1 + c_2 + 2d_1\right)F^{(T)}_{\mu\nu}F^{(T)\mu\nu} - \frac{1}{72}\left(4c_1 + c_2 + d_1\right)F^{(S)}_{\mu\nu}F^{(S)\mu\nu} + \frac{1}{24}\left(c_2 - 2c_1\right)\nabla_{\mu}S^{\mu}\nabla_{\nu}S^{\nu}\\
&+ \frac{1}{54}\left[2c_1 - 4c_2 - 9\left(4h_3 + 6h_{13} + h_{14}\right)\right]S^{\mu}S^{\nu}\nabla_{\mu}T_{\nu} - 2h_2T_{\mu}T^{\mu}\nabla_{\nu}T^{\nu} - \frac{2}{3}\left(h_{14} - h_{15}\right)\varepsilon^{\lambda\rho\mu\nu}T_{\lambda}S_{\rho}\partial_{\mu}T_{\nu}\\
&+ \frac{1}{18}\left[2c_1 - c_2 - 3\left(6h_{13} + h_{14} - h_{15}\right)\right]T^{\mu}S^{\nu}\nabla_{\nu}S_{\mu} + h_1G_{\mu\nu}T^{\mu}T^{\nu} + \frac{1}{2}\left(h_1 + 2h_2\right)RT_{\mu}T^{\mu}\\
&- \frac{1}{108}\left[4c_1 + c_2 + 9\left(4h_3 + 24h_4 + 2h_{14} - h_{15}\right)\right]S_{\mu}S^{\mu}\nabla_{\nu}T^{\nu} + \frac{1}{36}\left(4c_1 + c_2 + 36h_3\right)G_{\mu\nu}S^{\mu}S^{\nu}\\
&+ \frac{1}{72}\left[4c_1 + c_2 + 36\left(h_3 + 2h_4\right)\right]RS_{\mu}S^{\mu} + \frac{1}{2}m_T^2T_{\mu}T^{\mu} + \frac{1}{2}m_S^2S_{\mu}S^{\mu} - \frac{2}{3}h_2T_{\mu}T^{\mu}T_{\nu}T^{\nu}\\
&- \frac{1}{648}\left[16c_1 + 4c_2 - 9\left(h_1 + 3h_2 - 16h_3 - 48h_4 - 8h_{14}\right)\right]T_{\mu}T^{\mu}S_{\nu}S^{\nu} + \frac{1}{24}h_4S_{\mu}S^{\mu}S_{\nu}S^{\nu}\\
&+ \frac{1}{648}\left[16c_2 - 8c_1 - 9\left(h_1 - 16h_3 - 48h_{13} - 8h_{14}\right)\right]T_{\mu}S^{\mu}T_{\nu}S^{\nu}\,.\end{aligned} \tag{13}$$

- The term $\varepsilon^{\lambda\rho\mu\nu}T_{\lambda}S_{\rho}\partial_{\mu}T_{\nu}$ does not involve higher order time derivatives in the field equations, while $T^2\nabla T$ presents a safe Galileon-like derivative coupling, **thus representing Ostrogradsky-free interactions for the vector and axial modes**.
- Conversely, the terms of the form $(\nabla S)^2$, $S^2\nabla T$, $TS\nabla S$ and RS^2, driven by the axial mode, excite the longitudinal components of the modes with higher time derivatives in the field equations, **thus representing Ostrogradsky instabilities around any general background**.
- An exception is for the terms $G_{\mu\nu}T^{\mu}T^{\nu}$ and $G_{\mu\nu}S^{\mu}S^{\nu}$, which do not involve higher order time derivatives and Ostrogradsky instabilities due to the divergencelessness of the Riemannian Einstein tensor.
- Likewise, **ghost-free kinetic terms are guaranteed by fixing negative signs**.
- Overall, **the avoidance of ghostly and Ostrogradsky instabilities requires to set the following conditions between the Lagrangian coefficients**:

$$c_2 = 2c_1\,, \quad h_2 = -\frac{h_1}{2}\,, \quad h_3 = -\frac{1}{6}\left(c_1 + 6h_{13}\right)\,, \quad h_4 = \frac{h_{13}}{2}\,, \quad h_{14} = -2h_{13}\,, \quad h_{15} = 4h_{13}\,, \tag{14}$$

$$d_1 \leqslant 0\,, \quad -\frac{d_1}{6} \leqslant c_1 \leqslant -\frac{d_1}{3}\,. \tag{15}$$

4. Adding nonmetricity in the stability analysis

- The inclusion of nonmetricity in the stability analysis **demands the presence of cubic order invariants involving the two vector modes of this tensor**.
- For the Weyl vector, which is associated with the trace of the tensor, **the resulting Lagrangian develops Ostrogradsky instabilities** due to terms of the form $S^2\nabla W$, $T^2\nabla W$, $W^2\nabla T$, $SW\nabla S$, $TW\nabla T$, $TW\nabla W$, RW^2 and RTW.
- The second vector, which is associated with the traceless part of the tensor, **also includes pathological terms** of the form $(\nabla\Lambda)^2$, $\nabla T\nabla\Lambda$, $T^2\nabla\Lambda$, $S^2\nabla\Lambda$, $\Lambda^2\nabla T$, $T\Lambda\nabla T$, $T\Lambda\nabla\Lambda$, $S\Lambda\nabla S$, $R\Lambda^2$, $R\nabla\Lambda$ and $RT\Lambda$.
- Furthermore, there are cubic interactions between the vector modes of torsion and nonmetricity, described in the Lagrangian by terms of the form $T\Lambda\nabla W$ and $W\Lambda\nabla T$, **which also constitute Ostrogradsky instabilities**.
- **But a more exhaustive fine-tuning of the Lagrangian coefficients allows all those instabilities to be suppressed**.
- The general setup also demands to describe the kinetic terms by means of a kinetic matrix

$$\mathcal{L}_{FF} = -\frac{1}{4}\kappa_{XY}F^{(X)}_{\mu\nu}F^{(Y)\mu\nu}\,, \quad F^{(X)} = \left(F^{(S)}, F^{(T)}, F^{(W)}, F^{(\Lambda)}\right)\,, \tag{16}$$

where

$$\kappa_{XY} = \begin{pmatrix}\kappa_{SS} & 0 & 0 & 0\\ 0 & \kappa_{TT} & \kappa_{TW} & \kappa_{T\Lambda}\\ 0 & \kappa_{TW} & \kappa_{WW} & \kappa_{W\Lambda}\\ 0 & \kappa_{T\Lambda} & \kappa_{W\Lambda} & \kappa_{\Lambda\Lambda}\end{pmatrix}\,. \tag{17}$$

- The requirement of ghost-free kinetic terms can be then implemented by **fixing first the appropriate signs for the eigenvalues of the kinetic matrix in Weyl-Cartan geometry, and combining the resulting conditions with those provided by the inclusion of the vector mode of the traceless nonmetricity tensor**.
- Such a procedure splits the general kinetic matrix as

$$\kappa_{XY} = \kappa_{XY}^{\rm Weyl-Cartan} + \kappa_{XY}^{\rm Extra} = \begin{pmatrix}\kappa_{SS} & 0 & 0 & 0\\ 0 & \kappa_{TT} & \kappa_{TW} & 0\\ 0 & \kappa_{TW} & \kappa_{WW} & 0\\ 0 & 0 & 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & \kappa_{T\Lambda}\\ 0 & 0 & 0 & \kappa_{W\Lambda}\\ 0 & \kappa_{T\Lambda} & \kappa_{W\Lambda} & \kappa_{\Lambda\Lambda}\end{pmatrix}\,, \tag{18}$$

with $X, Y = S, T, W, \Lambda$, in such a way that **the semidefinite positivity of $\kappa_{XY}^{\rm Weyl-Cartan}$ and $\kappa_{XY}^{\rm Extra}$ ensures that the eigenvalues of the general kinetic matrix are nonnegative**.

- In summary, while in quadratic MAG the dynamics of the vector and axial modes of torsion and nonmetricity is plagued of instabilities, **all of them satisfy the stability requirements in the framework of cubic MAG**.

5. Reissner-Nordström-like black holes with dynamical torsion and nonmetricity

- We study the field equations of cubic MAG in a static and spherically symmetric space-time

$$ds^2 = \Psi_1(r)dt^2 - \frac{dr^2}{\Psi_2(r)} - r^2d\vartheta^2 - r^2\sin\vartheta^2d\varphi^2\,. \tag{19}$$

- The metric, torsion and nonmetricity tensors are then preserved under time translations and spatial rotations

$$\xi_0 = \partial_t\,,\, \xi_3 = \sqrt{1 - kr^2}\left(\cos\vartheta\,\partial_r - \frac{1}{r}\sin\vartheta\,\partial_{\vartheta}\right)\,, \tag{20}$$

$$\begin{aligned}\xi_1 &= \sqrt{1 - kr^2}\left(\sin\vartheta\cos\varphi\,\partial_r + \frac{1}{r}\cos\vartheta\cos\varphi\,\partial_{\vartheta} - \frac{\sin\varphi}{r\sin\vartheta}\partial_{\varphi}\right)\,,\\
\xi_2 &= \sqrt{1 - kr^2}\left(\sin\vartheta\sin\varphi\,\partial_r + \frac{1}{r}\cos\vartheta\sin\varphi\,\partial_{\vartheta} + \frac{\cos\varphi}{r\sin\vartheta}\partial_{\varphi}\right)\,.\end{aligned} \tag{21}$$

- In line with quadratic MAG, the resulting field equations also admit **Reissner-Nordström-like black hole solutions with dynamical torsion and nonmetricity**:

$$\Psi(r) \equiv \Psi_1(r) = \Psi_2(r) = 1 - \frac{2m}{r} + \frac{\alpha_1\kappa_{\rm s}^2 + \alpha_2\kappa_{\rm d}^2 + \alpha_3\kappa_{\rm sh}^2}{r^2}\,, \tag{22}$$

- But in contrast with quadratic MAG, the cubic MAG model of the solutions incorporate massive tensor modes, **which guarantees the avoidance of further strong coupling problems and no-go theorems that constrain the interaction of such modes in the quantum regime**.