

Cosmological Perturbation Theory in Metric-Affine Gravity

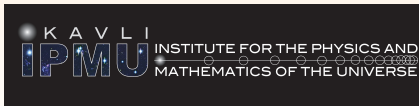
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Jointly with Jorge Gigante Valcarcel, K.Aoki, and M.A. Gorji.



Post-Riemannian decomposition

- In the most general metric-affine setting, the fundamental variables are a **metric** $g_{\mu\nu}$ (10 comp.) as well as the coefficients $\tilde{\Gamma}^{\rho}_{\mu\nu}$ (64 comp.) of an **affine connection**.

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- Then, torsion and nonmetricity are defined as

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- Further, we can write the curvature as:

$$\begin{aligned}\tilde{\Gamma}^\lambda_{\mu\nu} &= \Gamma^\lambda_{\mu\nu} + K^\lambda_{\mu\nu} + L^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + N^\lambda_{\mu\nu} \\ \tilde{R}^\lambda_{\rho\mu\nu} &= R^\lambda_{\rho\mu\nu} + 2\nabla_{[\mu} N^\lambda_{\rho|\nu]} + 2N^\lambda_{\sigma[\mu} N^\sigma_{\rho|\nu]}.\end{aligned}$$

3 + 1 decomposition - settling notation

- Let us first introduce a projector tensor

$$P_{\mu\nu} \equiv g_{\mu\nu} + n_{\mu}n_{\nu} ,$$

which is orthogonal ($P_{\mu\nu}n^{\mu} = 0$) to a timelike vector n_{μ} satisfying $n_{\mu}n^{\mu} = -1$.

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- Let us introduce arrows on top of quantities to denote traceless-spatial and symmetric quantities, i.e.:

$$\vec{X}_{\mu_1 \dots \mu_n} \equiv P_{\mu_1}^{\alpha_1} \dots P_{\mu_n}^{\alpha_n} X_{\alpha_1 \dots \alpha_n},$$

$$\vec{X}_{\mu_1 \dots \mu_n} = \vec{X}_{(\mu_1 \dots \mu_n)},$$

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- One can then straightforwardly decompose scalar and vectors:

$$\phi = \phi, \quad X_\mu = \vec{X}_\mu - n_\mu X_*.$$

Cosmology in MAG

- Let us assume that the metric, torsion and nonmetricity have the same cosmological symmetries (isotropy and homogeneity)

$$\mathcal{L}_\xi \bar{g}_{\mu\nu} = \mathcal{L}_\xi \bar{T}^\lambda{}_{\mu\nu} = \mathcal{L}_\xi \bar{Q}^\lambda{}_{\mu\nu} = 0.$$

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- By solving those equations one get the FLRW metric and torsion+nonmetricity satisfying homogeneity and isotropy:

$$\begin{aligned}\bar{g} &= -\bar{n}_\mu \bar{n}_\nu dx^\mu \otimes dx^\nu + \bar{P}_{\mu\nu} dx^\mu \otimes dx^\nu = -N^2 dt \otimes dt + a^2 \gamma_{ij} dx^i \otimes dx^j, \\ \bar{T}^\lambda{}_{\mu\nu} &= 2T_1(t) \bar{n}_{[\mu} \bar{P}_{\nu]}{}^\lambda + 2T_2(t) \bar{\varepsilon}^\lambda{}_{\mu\nu\rho} \bar{n}^\rho, \\ \bar{Q}^\lambda{}_{\mu\nu} &= 2Q_1(t) \bar{n}_\lambda \bar{n}_\mu \bar{n}_\nu + 2Q_2(t) \bar{n}_\lambda \bar{P}_{\mu\nu} + 2Q_3(t) \bar{P}_{\lambda(\mu} \bar{n}_{\nu)},\end{aligned}$$

where $\gamma_{ij} dx^i \otimes dx^j = \frac{dr^2}{1-Kr^2} + r^2 d\Omega^2$. Note that there are 5 independent functions coming from Post-Riemannian.

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- Our aim is to perform a 3 + 1+SVT-decomposition for all the tensors in MAG up to linear perturbations to then write:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}, \quad \bar{T}^\lambda{}_{\mu\nu} = \bar{T}^\lambda{}_{\mu\nu} + \delta T^\lambda{}_{\mu\nu}, \quad \bar{Q}^\lambda{}_{\mu\nu} = \bar{Q}^\lambda{}_{\mu\nu} + \delta Q^\lambda{}_{\mu\nu}.$$

Cosmology in MAG - very short discussion

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- Usually, it is possible to avoid cosmological singularities and replace them a cosmic bounce.
- Dark energy can be explained by the scalar modes of torsion.
- It is possible to find inflationary models such as Einstein-Cartan couple to Higgs or other more complicated ones which are compatible with observations.

3 + 1 decomposition for FLRW - metric sector

- A symmetric rank-2 tensor (as the metric) can be easily decomposed as 1 tensor, 1 vector and 2 scalars:

$$X_{(\mu\nu)} = n_\mu n_\nu X_{**} + \frac{1}{3} P_{\mu\nu} X_{***} + \vec{X}_{\mu\nu} - 2n_{(\mu} \vec{X}_{*\nu)}.$$

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- Then, the metric perturbation can be decomposed as

$$\delta g_{\mu\nu} = -\bar{n}_\mu \bar{n}_\nu \alpha + \bar{P}_{\mu\nu} \zeta + \vec{h}_{\mu\nu} - 2\bar{n}_{(\mu} \vec{\beta}_{\nu)},$$

in such a way that the full metric tensor is then given by

$$\bar{g}_{\mu\nu} + \delta g_{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} -N^2(1 + \alpha) & aN\vec{\beta}_i \\ aN\vec{\beta}_j & a^2[(1 + \zeta)\gamma_{ij} + \vec{h}_{ij}] \end{pmatrix}.$$

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- First of all, it is useful to use a pseudo-orthogonal decomposition for torsion as (in 4 dimensions)

$$T^\lambda{}_{\mu\nu} = \frac{1}{3} \left(\delta^\lambda{}_\nu T_\mu - \delta^\lambda{}_\mu T_\nu \right) + \frac{1}{6} \varepsilon^\lambda{}_{\rho\mu\nu} S^\rho + t^\lambda{}_{\mu\nu}.$$

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 - $t_{\lambda\mu\nu}$ carries 16 dof since it is a rank-3 tensor antisymmetric in the last two indices, totally traceless and satisfying a cyclic condition $t_{[\lambda\mu\nu]} = 0$. \implies **This quantity is not so trivial to decompose**

3 + 1 decomposition for FLRW - Torsion sector

- The irreducible modes of antisymmetric rank-3 tensor (as torsion) can be decomposed as

$$T_\mu = \vec{T}_\mu - n_\mu \phi,$$

$$S_\mu = \vec{S}_\mu - n_\mu \varrho,$$

$$t_{\mu\nu\rho} = 2n_{[\nu}\vec{A}_{\rho]\mu} + 2\left(n_\mu n_{[\nu} + \frac{1}{2}P_{\mu[\nu}\right)\vec{B}_{\rho]} + \frac{1}{2}\varepsilon_{\nu\rho}{}^{\alpha\beta}\left[-n_\alpha\vec{A}_{\beta\mu} + \left(n_\beta n_\mu + \frac{1}{2}P_{\beta\mu}\right)\vec{B}_\alpha\right].$$

where we recall that the quantities $\vec{A}_{\mu\nu}$ and $\vec{A}_{\mu\nu}$ with an arrow on top are spatial, symmetric and traceless.

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- Then, torsion is decomposed as

$$T^\lambda{}_{\mu\nu} = \bar{T}^\lambda{}_{\mu\nu} + \delta T^\lambda{}_{\mu\nu} = \begin{cases} T^0{}_{0i} = -\frac{a}{3} (\vec{T}_i + 3\vec{B}_i) \\ T^0{}_{ij} = -\frac{a^2}{6N} \varepsilon_{ijk} (\vec{S}^k + 3\vec{B}^k) \\ T^i{}_{0j} = -N \left[\vec{A}^i{}_j - \frac{1}{3} \delta^i{}_j (\bar{\phi} + \phi) + \frac{1}{6} \varepsilon^i{}_{jk} (\vec{S}^k - \frac{3}{2} \vec{B}^k) \right] \\ T^i{}_{jk} = -\frac{a}{6} \left\{ 4\delta^i{}_{[j} \vec{T}_{k]} - 6\delta^i{}_{[j} \vec{B}_{k]} + \left[3\varepsilon_{jkl} \vec{A}^{il} - \varepsilon^i{}_{jk} (\bar{\varrho} + \varrho) \right] \right\} \end{cases},$$

where we have introduced the background T_i in the scalars ϕ, ϱ .

3 + 1 decomposition for FLRW - Nonmetricity sector

- Similarly as we did before, one can decompose nonmetricity in the pseudo-orthogonal group as (in 4 dimensions)

$$Q_{\lambda\mu\nu} = g_{\mu\nu}W_\lambda + \mathcal{Q}_{\lambda\mu\nu},$$

$$\mathcal{Q}_{\lambda\mu\nu} = \frac{1}{2}(g_{\lambda\mu}\Lambda_\nu + g_{\lambda\nu}\Lambda_\mu) - \frac{1}{4}g_{\mu\nu}\Lambda_\lambda + \frac{1}{3}\varepsilon_{\lambda\rho\sigma(\mu}\Omega_{\nu)}{}^{\rho\sigma} + q_{\lambda\mu\nu}.$$

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- The irreducible modes of symmetric rank-3 tensor (as nonmetricity) can be then decomposed as

$$W_\mu = \vec{W}_\mu - n_\mu \theta,$$

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$$\Omega_{\mu\nu\rho} = 2n_{[\nu} \vec{Q}_{\rho]\mu} + 2\left(n_\mu n_{[\nu} + \frac{1}{2}P_{\mu[\nu}\right) \vec{Y}_{\rho]} + \frac{1}{2}\varepsilon_{\nu\rho}{}^{\alpha\beta} \left[-n_\alpha \vec{Q}_{\beta\mu} + \left(n_\beta n_\mu + \frac{1}{2}P_{\beta\mu}\right) \vec{Y}_\alpha\right],$$

$$q_{\mu\nu\rho} = \vec{C}_{\mu\nu\rho} - 3n_{(\mu} \vec{\kappa}_{\nu\rho)} + \frac{3}{5}(5n_{(\mu} n_\nu + P_{(\mu\nu}) \vec{Z}_{\rho)} - (n_{(\mu} n_\nu + P_{(\mu\nu}) n_\rho) \xi.$$

where $\vec{C}_{\mu\nu\rho}$, $\vec{\kappa}_{\mu\nu}$, $\vec{Q}_{\mu\nu}$ and $\vec{Q}_{\mu\nu}$ are spatial, symmetric and traceless tensors.

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- Then, nonmetricity is then decomposed as

$$Q^\lambda{}_{\mu\nu} = \begin{cases} Q_{000} = -N^3 \left[\bar{\theta} + \theta + \frac{3}{4}(\bar{\psi} + \psi) - (\bar{\xi} + \xi) \right] \\ Q_{00i} = -\frac{N^2 a}{2} \left(\bar{\Lambda}_i - \frac{1}{3} \bar{Y}_i - 2\bar{Z}_i \right) \\ Q_{0ij} = -\frac{Na^2}{3} \left\{ \bar{Q}_{ij} - 3\bar{\kappa}_{ij} - 3 \left[\bar{\theta} + \theta - \frac{1}{4}(\bar{\psi} + \psi) + \frac{1}{3}(\bar{\xi} + \xi) \right] \gamma_{ij} \right\} \\ Q_{i00} = -N^2 a \left(\bar{W}_i - \frac{1}{4} \bar{\Lambda}_i + \frac{1}{3} \bar{Y}_i - \bar{Z}_i \right) \\ Q_{i0j} = \frac{Na^2}{6} \left\{ [\bar{Q}_{ij} + 6\bar{\kappa}_{ij} + (3(\bar{\psi} + \psi) + 2(\bar{\xi} + \xi)) \gamma_{ij}] - 3\varepsilon_{ijk} \bar{Y}^k \right\} \\ Q_{ijk} = a^3 \left\{ \bar{C}_{ijk} + \frac{1}{4}(4\bar{W}_i - \bar{\Lambda}_i) \gamma_{jk} + \gamma_{i(j} \bar{\Lambda}_{k)} - \frac{1}{6}(\gamma_{jk} \bar{Y}_i - \gamma_{i(j} \bar{Y}_{k)}) + \frac{3}{5} \gamma_{(ij} \bar{Z}_{k)} + \frac{2}{3} \varepsilon_{li(j} \bar{Q}^l{}_{k)} \right\} \end{cases}$$

where we have introduced the background Q_i in the scalars θ, ψ, ξ .

Summary of modes

- We can group the modes as:

$$\delta\vec{\mathbf{X}} = \{\alpha, \zeta, \phi, \varrho, \theta, \psi, \xi\},$$

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Spin and Parity	0 ⁺	0 ⁻	1 ⁺	1 ⁻	2 ⁺	2 ⁻	3 ⁻
Metric sector $g_{\mu\nu}$	α, ζ	-	-	$\vec{\beta}_i$	\vec{h}_{ij}	-	-
Torsion sector $T^\lambda{}_{\mu\nu}$	ϕ	ϱ	$\vec{S}_i, \vec{\mathcal{B}}_i$	\vec{T}_i, \vec{B}_i	\vec{A}_{ij}	$\vec{\mathcal{A}}_{ij}$	-
Nonmetricity sector $Q_{\lambda\mu\nu}$	θ, ψ, ξ	-	$\vec{\mathcal{Y}}_i$	$\vec{W}_i, \vec{\Lambda}_i, \vec{Y}_i, \vec{Z}_i$	$\vec{Q}_{ij}, \vec{\kappa}_{ij}$	$\vec{\mathcal{Q}}_{ij}$	\vec{C}_{ijk}

Table: Species in MAG. K. Aoki, S. Bahamonde, J. Gigante Valcarcel and M. A. Gorji, "Cosmological Perturbation Theory in Metric-Affine Gravity," [arXiv:2310.16007 [gr-qc]].

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Here, the superscript “(T)” refers that the quantity is transverse, while the tensors denoted by the superscript “(TT)” are transverse-traceless and symmetric:

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- New rank-3 part:** Note that the SVT decomposition of a spatially rank-3 symmetric and traceless tensor does not appear in standard Riemannian curvature theories.

- **Metric sector:** the 10 dof are split in terms of four scalars $\{\alpha, \beta, \zeta, h\}$ (1 dof each), two transverse vectors $\{\beta_i^{(T)}, h_i^{(T)}\}$ (2 dof each), and one symmetric and transverse-traceless tensor $h_{ij}^{(TT)}$ (2 dof).

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SVT	Quantities	dof	Total dof
5 scalars	$\{T, B, \phi, A, \varrho\}$	1 dof each	5
3 pseudoscalars	$\{\mathcal{S}, \mathcal{B}, \mathcal{A}\}$	1 dof each	3
3 vectors	$\{T_i^{(T)}, B_i^{(T)}, A_i^{(T)}\}$	2 dof each	6
3 pseudovectors	$\{\mathcal{S}_i^{(T)}, \mathcal{B}_i^{(T)}, \mathcal{A}_i^{(T)}\}$	2 dof each	6
1 rank-2 tensor	$\{A_{ij}^{(TT)}\}$	2 dof each	2
1 rank-2 pseudotensor	$\{\mathcal{A}_{ij}^{(TT)}\}$	2 dof each	2

Table: Perturbation spectrum for the torsion tensor. K. Aoki, S. Bahamonde, J. Gigante Valcarcel and M. A. Gorji, "Cosmological Perturbation Theory in Metric-Affine Gravity," [arXiv:2310.16007 [gr-qc]].

SVT	Quantities	dof	Total dof
10 scalars	$\{\theta, \psi, \xi, \Lambda, Y, Z, \kappa, Q, W, C\}$	1 dof each	10
2 pseudoscalars	$\{\mathcal{Y}, \mathcal{Q}\}$	1 dof each	2
7 vectors	$\{\Lambda^{(T)}_i, Y^{(T)}_i, Z^{(T)}_i, \kappa^{(T)}_i, Q^{(T)}_i, W^{(T)}_i, C^{(T)}_i\}$	2 dof each	14
2 pseudovectors	$\{\mathcal{Y}^{(T)}_i, \mathcal{Q}^{(T)}_i\}$	2 dof each	4
3 rank-2 tensor	$\{\kappa^{(TT)}_{ij}, Q^{(TT)}_{ij}, C^{(TT)}_{ij}\}$	2 dof each	6
1 rank-2 pseudotensor	$\{\mathcal{Q}^{(TT)}_{ij}\}$	2 dof each	2
1 rank-3 tensor	$\{C^{(TTT)}_{ijk}\}$	2 dof each	2

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1 rank-3 tensor	$\{C^{(TTT)}_{ijk}\}$	2 dof each	2

Table: Perturbation spectrum for the nonmetricity tensor.

- Note that the spin-3 carries a helicity-3 part $\{C^{(TTT)}_{ijk}\}$ and since the metric does not have such terms, it totally decouples from the other modes in FLRW.

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- On the other hand, the spatially homogeneous and isotropic form of the hypermomentum tensor reads (D. Iosifidis, Eur. Phys. J. C **80** (2020) no.11, 1042):

$$\bar{\Delta}_{\lambda\mu\nu} = \frac{1}{3} \bar{\Delta}_1(t) \bar{P}_{\lambda\mu} \bar{n}_\nu + \bar{\Delta}_2(t) \bar{P}_{\lambda\nu} \bar{n}_\mu + \bar{\Delta}_3(t) \bar{n}_\lambda \bar{P}_{\mu\nu} + \frac{1}{4} \bar{\Delta}_4(t) \bar{n}_\lambda \bar{n}_\mu \bar{n}_\nu + \bar{\Delta}_5(t) \bar{\varepsilon}_{\lambda\mu\nu\rho} \bar{n}^\rho ,$$

Matter content: Background

- We assume that the matter content satisfies the cosmological principle.
- The corresponding metric energy-momentum tensor acquires the form of a perfect fluid

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- Hypermomentum has five arbitrary functions $\{\bar{\Delta}_i\}_{i=1}^5$ representing the **intrinsic spin, dilation and shear currents of matter**.

Matter content: Perturbations

- Considering general perturbations around both matter sectors:

$$\Theta_{\mu\nu} = \bar{\Theta}_{\mu\nu} + \delta\Theta_{\mu\nu}, \quad \Delta_{\mu\nu\lambda} = \bar{\Delta}_{\mu\nu\lambda} + \delta\Delta_{\mu\nu\lambda},$$

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$$\delta\Theta_{\mu\nu} \longrightarrow \begin{cases} \delta\Theta_{00} = N^2\delta\rho - \bar{\rho}\delta g_{00} \\ \delta\Theta_{0i} = -N(\bar{\rho} + \bar{p})\delta\vec{u}_i + \bar{p}\delta g_{0i} \\ \delta\Theta_{ij} = \bar{p}\delta g_{ij} + a^2(\gamma_{ij}\delta p + \bar{\pi}_{ij}) \end{cases},$$

with $\delta\rho$ and δp are the perturbations in the energy density and pressure; $\delta\vec{u}_i$ is the spatial velocity vector and $\bar{\pi}_{ij}$ is the spatial stress tensor.

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$$\delta\vec{u}_i = D_i\delta u + \delta u_i^{(T)},$$
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- Hypermomentum:**

- It can introduce up to 64 d.o.f. with the SVT decomposition behaving in the same way as the sum of the perturbation of torsion (24 dof) and nonmetricity (40 dof) as we described before.

Fourier space

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- For scalars, we have

$$\mathbf{S}(t, \vec{x}) = \frac{1}{(2\pi)^3} \int d^3k \frac{1}{\sqrt{2}} \mathbf{S}(t, \vec{k}) E(\vec{x}; \vec{k}) + \text{c.c.},$$

with real functions $\mathbf{S}(t, \vec{k})$ where E are the orthogonal eigenstates of γ_{ij} :

$$D^2 E = -k^2 E, \quad \int dx^3 \sqrt{\gamma} E^*(\vec{x}; \vec{k}) E(\vec{x}; \vec{k}') = (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}'),$$

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- The helicity-1, 2, and 3 modes can be similarly mapped to the momentum space but each helicity still contains two different modes.
- In MAG, it is particularly convenient to characterise them by using the circular polarisation bases $E_i^{(A)}$, $E_{ij}^{(A)}$, and $E_{ijk}^{(A)}$ ($A = L, R$); they are defined by the eigenstates of the operators

$$\begin{aligned} D^2 E_i^{(A)} &= -k^2 E_i^{(A)}, & \varepsilon_i{}^{pq} D_p E_q^{(A)} &= \lambda_A k E_i^{(A)}, \\ D^2 E_{ij}^{(A)} &= -k^2 E_{ij}^{(A)}, & \varepsilon_{(i}{}^{pq} D_p E_{q|j)}^{(A)} &= \lambda_A k E_{ij}^{(A)}, \\ D^2 E_{ijk}^{(A)} &= -k^2 E_{ijk}^{(A)}, & \varepsilon_{(i}{}^{pq} D_p E_{q|jk)}^{(A)} &= \lambda_A k E_{ijk}^{(A)}, \end{aligned}$$

with $\lambda_L = -1$, $\lambda_R = +1$, and satisfy orthogonality conditions (similar than the scalar part).

Fourier space

- Then, the corresponding Fourier transformations are

$$\mathbf{S}(t, \vec{x}) = \frac{1}{(2\pi)^3} \int d^3k \frac{1}{\sqrt{2}} \mathbf{S}(t, \vec{k}) E(\vec{x}; \vec{k}) + \text{c.c.},$$

$$\mathcal{S}(t, \vec{x}) = \frac{1}{(2\pi)^3} \int d^3k \frac{1}{\sqrt{2}} \mathcal{S}(t, \vec{k}) E(\vec{x}; \vec{k}) + \text{c.c.},$$

$$\mathbf{V}_i^{(T)}(t, \vec{x}) = \frac{1}{(2\pi)^3} \int d^3k \frac{1}{\sqrt{2}} [\mathbf{V}^{(L)}(t, \vec{k}) E_i^{(L)}(\vec{x}; \vec{k}) + \mathbf{V}^{(R)}(t, \vec{k}) E_i^{(R)}(\vec{x}; \vec{k})] + \text{c.c.},$$

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- Flat FLRW case** Let us consider the circular polarisation bases with $K = 0$ in the Cartesian metric $\gamma_{ij} = \delta_{ij}$:

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- The modes can be written as (for scalars, vectors and tensors)

$$E = e^{i\vec{k}\cdot\vec{x}}, \quad E_i^{(A)} = e_i^{(A)} e^{i\vec{k}\cdot\vec{x}}, \quad E_{ij}^{(A)} = e_i^{(A)} e_j^{(A)} e^{i\vec{k}\cdot\vec{x}}, \quad E_{ijk}^{(A)} = e_i^{(A)} e_j^{(A)} e_k^{(A)} e^{i\vec{k}\cdot\vec{x}},$$

with

$$e_i^{(L)} = (1/\sqrt{2}, -i/\sqrt{2}, 0), \quad e_i^{(R)} = (1/\sqrt{2}, i/\sqrt{2}, 0).$$

Application - Quadratic MAG model

- General quadratic parity preserving action with dynamical torsion and nonmetricity:

$$\begin{aligned}
 S = \int d^4x \sqrt{-g} \left\{ \mathcal{L}_m + \frac{1}{16\pi} \right. & \left[-\tilde{R} + a_1 \tilde{R}^2 + a_2 \tilde{R}_{\lambda\rho\mu\nu} \tilde{R}^{\lambda\rho\mu\nu} + a_3 \tilde{R}_{\lambda\rho\mu\nu} \tilde{R}^{\rho\lambda\mu\nu} \right. \\
 & + a_4 \tilde{R}_{\lambda\rho\mu\nu} \tilde{R}^{\mu\nu\lambda\rho} + a_5 \tilde{R}_{\lambda\rho\mu\nu} \tilde{R}^{\lambda\mu\rho\nu} + a_6 \tilde{R}_{\lambda\rho\mu\nu} \tilde{R}^{\mu\lambda\rho\nu} + a_7 \tilde{R}_{\rho\lambda\mu\nu} \tilde{R}^{\mu\lambda\rho\nu} \\
 & + a_8 \tilde{R}_{\mu\nu} \tilde{R}^{\mu\nu} + a_9 \tilde{R}_{\mu\nu} \tilde{R}^{\nu\mu} + a_{10} \hat{R}_{\mu\nu} \hat{R}^{\mu\nu} + a_{11} \hat{R}_{\mu\nu} \hat{R}^{\nu\mu} + a_{12} \tilde{R}_{\mu\nu} \hat{R}^{\mu\nu} \\
 & + a_{13} \tilde{R}_{\mu\nu} \hat{R}^{\nu\mu} + a_{14} \tilde{R}^\lambda{}_{\lambda\mu\nu} \tilde{R}^\rho{}_{\rho}{}^{\mu\nu} + a_{15} \tilde{R}^\lambda{}_{\lambda\mu\nu} \tilde{R}^{\mu\nu} + a_{16} \tilde{R}^\lambda{}_{\lambda\mu\nu} \hat{R}^{\mu\nu} \\
 & + b_1 T_{\lambda\mu\nu} T^{\lambda\mu\nu} + b_2 T_{\lambda\mu\nu} T^{\mu\lambda\nu} + b_3 T^\lambda{}_{\lambda\nu} T^\mu{}_{\mu}{}^\nu + c_1 T_{\lambda\mu\nu} Q^{\mu\lambda\nu} \\
 & + c_2 T^\lambda{}_{\lambda\nu} Q^{\nu\mu}{}_{\mu} + c_3 T^\lambda{}_{\lambda\nu} Q^{\mu\nu}{}_{\mu} + d_1 Q_{\lambda\mu\nu} Q^{\lambda\mu\nu} + d_2 Q_{\lambda\mu\nu} Q^{\mu\lambda\nu} \\
 & \left. + d_3 Q^\lambda{}_{\lambda\nu} Q^\mu{}_{\mu}{}^\nu + d_4 Q_{\nu}{}^\lambda{}_{\lambda} Q^{\nu\mu}{}_{\mu} + d_5 Q^\lambda{}_{\lambda\nu} Q^{\nu\mu}{}_{\mu} \right] \left. \right\} .
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- There are 27 coupling constants, but the above theory (in general) contains ghosts (see Jorge Gigante's talk for improvement).

Cosmological perturbations of the spin-3 field (helicity-3 part)

- Contribution of the spin-3 field (helicity-3 part) around flat FRW up to second order for the most general parity-preserving quadratic MAG action:

$$\begin{aligned} S_C^{(2)} &= \frac{1}{16\pi} \int dt d^3x a^3 \mathcal{G}_T^{(C)} \left[\left(\dot{C}_{ijk}^{(TT)} \right)^2 - \frac{1}{a^2} \left(D_l C_{ijk}^{(TT)} \right)^2 \right. \\ &\quad \left. + \frac{4}{a} T_2 \varepsilon_{ijk} C^{(TT)i}{}_{lm} D^j C^{(TT)klm} - m_{C,\text{eff}}^2 \left(C_{ijk}^{(TT)} \right)^2 \right] \\ &= \frac{1}{16\pi} \sum_{A=L,R} \int dt \frac{d^3k}{(2\pi)^3} a^3 \mathcal{G}_T^{(C)} \left[[\dot{C}^{(A)}]^2 - [\omega_{C,k}^{(A)}]^2 [C^{(A)}]^2 \right], \end{aligned}$$

where $[\omega_{C,k}^{(A)}]^2 \equiv \frac{k^2}{a^2} - 4\lambda_A T_2 \frac{k}{a} + m_{C,\text{eff}}^2$ and to avoid ghosts

$$\mathcal{G}_T^{(C)} = -\frac{1}{4}(2a_2 + 2a_3 + a_5 + a_6 + a_7) > 0.$$

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 &= \frac{1}{16\pi} \sum_{A=L,R} \int dt \frac{d^3k}{(2\pi)^3} a^3 \mathcal{G}_T^{(C)} \left[\dot{C}^{(A)}]^2 - [\omega_{C,k}^{(A)}]^2 [C^{(A)}]^2 \right],
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Recall here that $T_2 = T_2(t)$ is the torsion background function in the pseudo-vector S_μ .

- Field equation of the spin-3 field around flat FRW:

$$\ddot{C}^{(A)} + 3H\dot{C}^{(A)} + [\omega_{C,k}^{(A)}]^2 C^{(A)} = J^{(A)}.$$

- The quadratic action contains the parity-violating term

$$T_2 \varepsilon_{ijk} C^{(\text{TT})i}{}_{lm} D^j C^{(\text{TT})klm} \rightarrow \lambda_A T_2 k [C^{(A)}]^2 \quad (\text{momentum space}),$$

which originates from the covariant interaction $\varepsilon_{\alpha\nu\lambda\mu} q^{\rho\tau\nu} S^\alpha \nabla^\mu q_{\rho\tau}{}^\lambda$.

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- Note that if the background function $T_2 \neq 0$, we will always have such terms.

Conclusions

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- As an immediate application, we study linear perturbation of the nonmetricity helicity-3 modes (this model decouples from the metric) for a general parity-preserving action of metric-affine gravity which includes quadratic terms in curvature, torsion, and nonmetricity.

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- Perturbations of BH solutions: can we formulate a similar perturbation theory for black holes? quasinormal modes? Love Numbers?