Dynamical Systems: Cosmology and Biology

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Motivation

Dynamical systems are powerful

Different systems—from evolution of the Universe to epidemics—obey similar mathematical principles.

Let us explore how identical tools help us understand both.

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  - economic models, climate models, ...

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- Any solution $\psi(t)$ is called an *orbit* or *trajectory* in phase space.

Critical points

Critical (fixed, equilibrium) point

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- Physically: equilibrium configurations (e.g. steady populations, radiation–matter–dark energy balance, etc.).
- The key question: **what happens if we perturb** the system slightly away from $\mathbf{x}_0$?

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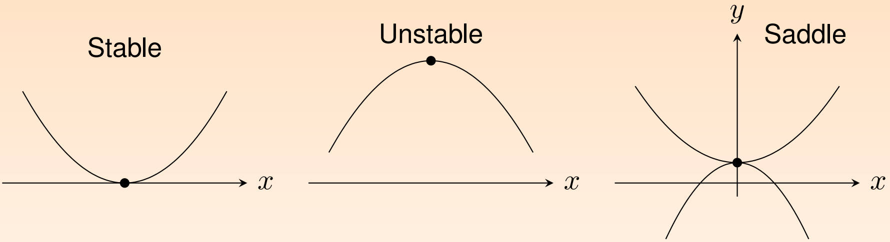
Saddle point

The fixed point is stable in some directions and unstable in others. Trajectories approach it along certain directions but move away along others.

Critical points – intuition



A ball in a potential landscape: valley (stable), hilltop (unstable), saddle (stable in one direction, unstable in another).

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- *Intuition: near equilibrium, many nonlinear systems behave approximately like linear ones (e.g. a pendulum near its lowest point behaves like a harmonic oscillator).*

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- If at least one eigenvalue has **zero** real part: $\Rightarrow$ the point is **non-hyperbolic** and linear theory is inconclusive.

2-dimensional dynamical system

- Consider a generic 2D system

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- We will see concrete examples later in cosmology and biology.

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- For, $k=1$ we say that the universe is spatially closed, for $k=-1$ it is spatially open and if $k=0$ it is spatially flat.
- The entire evolution of the Universe can be encoded in just one function of time, the scale factor $a(t)$. This makes cosmology particularly suitable for a dynamical–systems approach.

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Typical equation-of-state parameters

Component	w	Description
Radiation	$1/3$	significant pressure
Non-relativistic matter (dust)	0	negligible pressure
Dark energy	-1	$p < 0$: accelerated expansion

- By considering that the matter is described by a radiation term ($w = 1/3$) and a dust ($w = 0$), one gets the FLRW equations:

$$3H^2 = \kappa^2\rho_{\rm m} + \kappa^2\rho_{\rm r} + \Lambda\,, \quad 2\dot{H} + 3H^2 = -\frac{\kappa^2}{3}\rho_{\rm r} + \Lambda\,.$$

FLRW equations and Dynamical System: non-flat ΛCDM

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- Thanks to the assumptions $\rho_{\rm m} > 0$ and $\rho_{\rm r} > 0$ (positive energy), we obtain the constraints $x > 0$ and $y > 0$ which restrict the physical phase space in the (x, y)-plane Moreover, we get:

$$1 = x + y + \Omega_\Lambda\,.$$

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$$x = \Omega_{\rm m} = \frac{\kappa^2\rho_{\rm m}}{3H^2}, \quad y = \Omega_{\rm r} = \frac{\kappa^2\rho_{\rm r}}{3H^2} \quad \text{and} \quad \Omega_\Lambda = \frac{\kappa^2\rho_\Lambda}{3H^2}\,.$$

- Thanks to the assumptions $\rho_{\rm m} > 0$ and $\rho_{\rm r} > 0$ (positive energy), we obtain the constraints $x > 0$ and $y > 0$ which restrict the physical phase space in the (x, y)-plane Moreover, we get:

$$1 = x + y + \Omega_\Lambda\,.$$

- Because Ω_Λ is not independent, the dynamics of the entire cosmological model can be captured by the 2D system for (x, y). This makes the phase–space picture especially transparent.

- By doing some simple computations we can rewrite the flat FLRW as a dynamical system with $\eta = \log a$ ($d\eta = H dt$):

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Point	x	y	$w_{\rm eff}$	Eigenvalues	Stability
O	0	0	-1	$\{-4,-3\}$	Stable point
R	0	1	1/3	$\{1,4\}$	Unstable point
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Table: Critical points of the dynamical system and their properties.

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- In other words: the Universe naturally evolves from radiation domination $\rightarrow$ matter domination $\rightarrow$ dark–energy

This diagram shows how any initial combination of matter and radiation densities evolves over cosmic time.

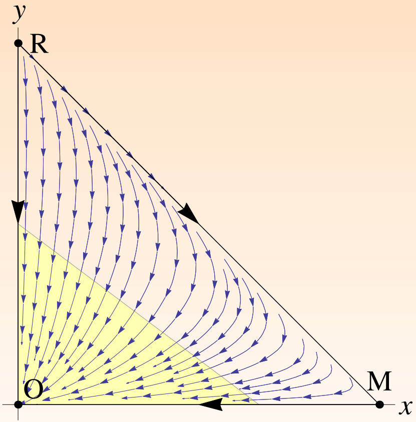

Figure: Phase space portrait of the dynamical system. The yellow/shaded area denotes the region of the phase space where the universe is accelerating.

Dark energy as a canonical scalar field

- Let us begin by considering a scalar field minimally coupled to gravity. The action which will then represent our physical system is

$$S = \int d^4x\sqrt{-g}\left(\frac{R}{2\kappa^2} + \mathcal{L}_{\rm m} - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right) \,,$$

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- Cosmological equations:

$$3H^2 = \kappa^2\left(\rho + \frac{1}{2}\dot{\phi}^2 + V\right)\,,$$

$$2\dot{H} + 3H^2 = -\kappa^2\left(w\rho + \frac{1}{2}\dot{\phi}^2 - V\right)\,,$$

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0\,.$$

- We can rewrite those equations as a dynamical system:

$$x' = -\frac{3}{2}\left[2x + (w-1)x^3 + x(w+1)\left(y^2-1\right) - \frac{\sqrt{2}}{\sqrt{3}}\lambda y^2\right] \,,$$
$$y' = -\frac{3}{2}y\left[(w-1)x^2 + (w+1)\left(y^2-1\right) + \frac{\sqrt{2}}{\sqrt{3}}\lambda x\right] \,,$$

with $1 = \Omega_{\rm m} + x^2 + y^2$ and we have defined

$$x = \frac{\kappa\dot{\phi}}{\sqrt{6}H} \quad y = \frac{\kappa\sqrt{V}}{\sqrt{3}H}\,,\,, \quad \lambda = -\frac{V_{,\phi}}{\kappa V}\,.$$

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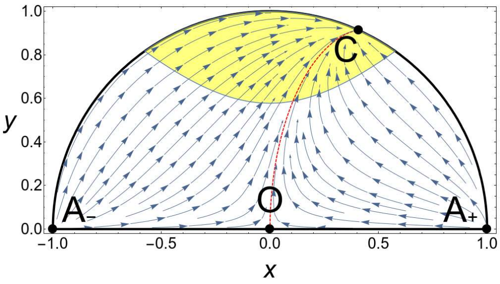

Outline

The SIR model

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- Total population is conserved:

$$N = S + I + R$$

SIR Differential Equations - Dynamical System

- The SIR model is given by:

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- Summing the equations gives population conservation:
  $(N = \text{total population})$

$$\frac{d}{dt}(S + I + R) = 0 \Rightarrow N = \text{constant}$$

SIR Modeling

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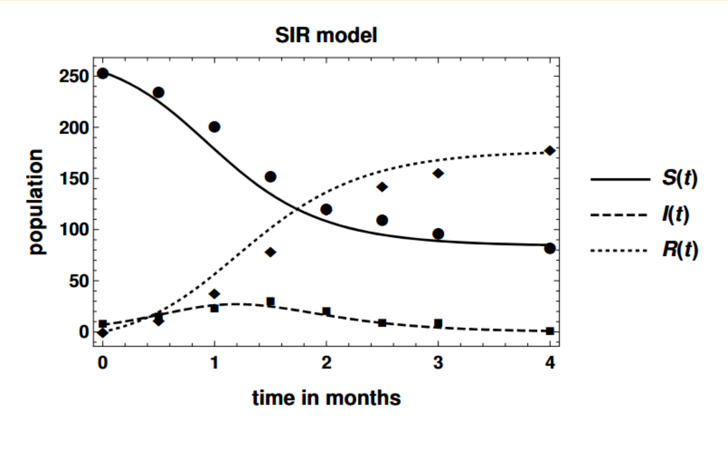
- The epidemic stops not because susceptibles run out, but because interactions $S \times I$ diminish.

Eyam Plague (1665)



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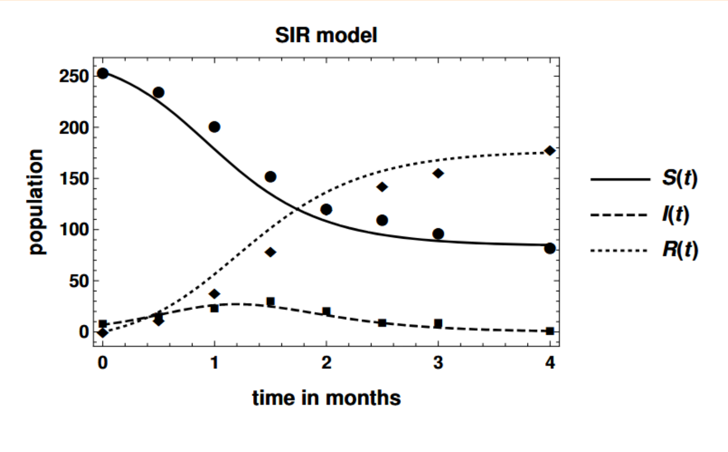

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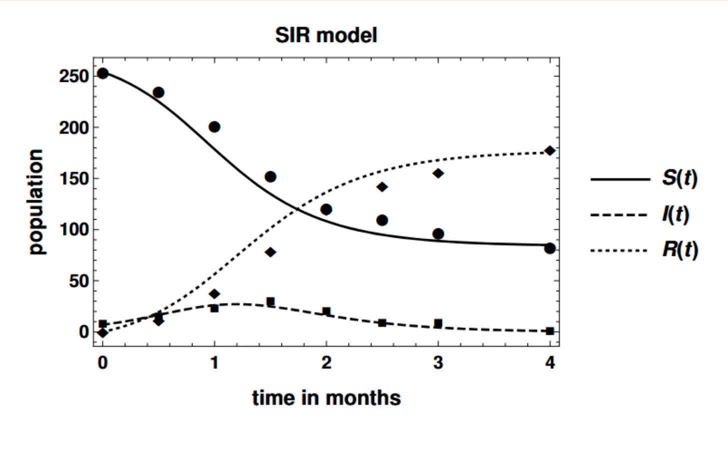

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- To model long-term diseases, demographic turnover must be included.

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- The population is no longer constant, but approaches $N(t) \to \dfrac{\Lambda}{\mu}$ as $t \to \infty$.
- We cannot solve this system analytically but we can write it as a 2D dynamical system (since $R = N - S - I$)

- The dynamical system can be written as:

$$x' = \rho(1 - x) - \mathcal{R}_0\, xy,$$
$$y' = (\mathcal{R}_0 x - 1)\, y,$$

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$$\rho = \frac{\mu}{(\beta + \mu)}, \qquad \mathcal{R}_0 = \frac{\Lambda\alpha}{\mu(\beta + \mu)}.$$

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- Will it die out, or will it become endemic (stay constantly in a specific population)?

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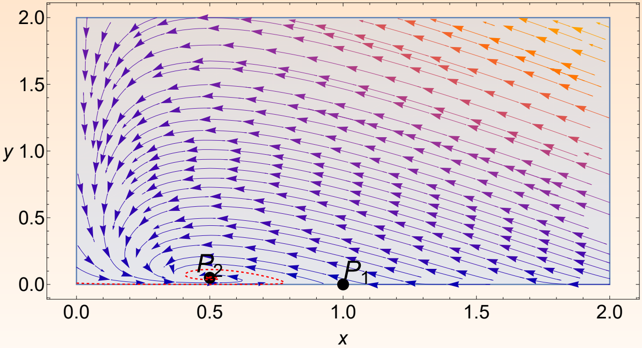

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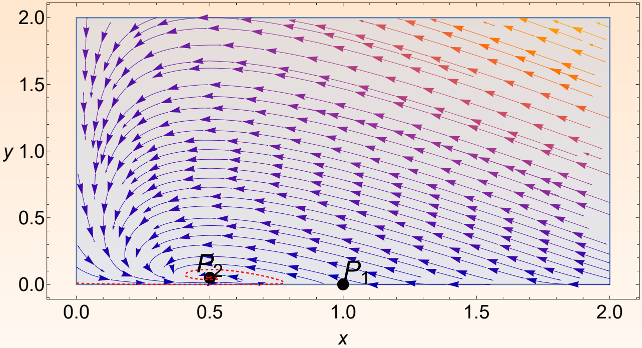

Trajectories converge to the disease-free state P_1 when $R_0 < 1$, and to the endemic state P_2 when $R_0 > 1$.

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- To study long-term **endemic diseases**, such as measles or chickenpox, where births replenish the susceptible population and maintain recurrent outbreaks.
- To analyze **population-level persistence** of infections, identifying conditions under which a disease becomes endemic (stable equilibrium) or dies out, using linear stability theory and reproduction numbers.

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- To study **reinfection dynamics** or partial loss of immunity, especially when demographic replacement interacts with waning immunity (decreasing immunity), producing multiple attractors.
- To evaluate the effect of **mortality and birth rates** on disease persistence, helping to determine thresholds for eradication in low-income or high-growth populations.

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Lotka–Volterra Predator–Prey Model

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- The fixed point P_2 is a center: nearby trajectories form closed periodic orbits.

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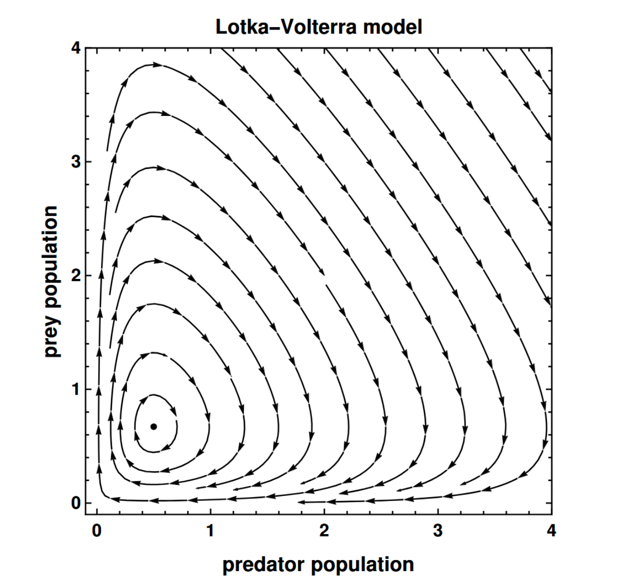
- $a > 0, c > 0$ control the natural growth and decay rates of predators and prey, respectively.
- $b > 0, d > 0$ model the effect that encounters have on the respective populations.
- Two critical points:
  - $P_1 = (x, y) = (0, 0)$: (no population whatsoever - not important)
  - $P_2 = (x, y) = (c/d, a/b)$: corresponds to the idealised state in which both populations are constant.
- The fixed point P_2 is a center: nearby trajectories form closed periodic orbits.
- One can show that the populations x and y vary cyclically or periodically.

Lotka–Volterra Predator–Prey Model



Outline

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- The same tools explain cosmic evolution (radiation $\rightarrow$ matter $\rightarrow$ dark energy) and biological dynamics (epidemics, ecological interactions).

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- **General question for the audience:**
  Where in your own research might a dynamical-systems perspective reveal hidden structure?